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نموذج رقم (١٨)
اقرار والتزام بالمعايير الأخلاقية والأمانة العلمية
وقوانين الجامعة الأردنية وأنظمتها وتعليماتها
لطلبة الماجستير

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عنوان الرسالة: ....

Pitman Closeness Comparison of Predictors of Future Order Statistics from Exponential Distribution

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PITMAN CLOSENESS COMPARISON OF PREDICTORS OF FUTURE ORDER STATISTICS FROM EXPONENTIAL DISTRIBUTION

By

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This Thesis was Submitted in Partial Fulfillment of the Requirements for the

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The University of Jordan

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DEDICATION

This thesis is respectfully dedicated to my:

great parents,

brothers, and sister,

my fiance',

all my friends,

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TABLE OF CONTENTS

All Rights Reserved - Library of University of Jordan - Center of Thesis Deposit

Chapter Three

The Comparisons between the Predictors under Pitman's Measure of Closeness

Chapter Four

Pitman Closeness Comparisons between Modified Predictor

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LIST OF TABLES

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List of symbol

δ_{BLUP} : The best linear unbiased predictor

δ_{BLIP} : The best linear invariant predictor

δ_{MLP} : The maximum likelihood predictor

δ_{CMP} : The conditional median predictor

δ_{MMLP} : The modified maximum likelihood predictor

π_0 : The mode of the order statistic sampled from Exp(1).

π_1 : The mean of the order statistic $Z_{s-r:n-r}$ sampled from Exp(1).

π_2 : The variance of the order statistic $Z_{s-r:n-r}$ sampled from Exp(1).

m: The median of the order statistic $Z_{s-r:n-r}$ sampled from Exp(1).

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ABSTRACT

In this thesis, we review the predictors of future order statistics from exponential distribution based on observed ordered data from doubly Type II censoring sample. We compare these predictors under Pitman's Measure of Closeness (PMC) and the pitman closeness probabilities are derived and computed for several cases. It is found that the conditional median predictor performs well when compared to each of the best linear unbiased predictor, the best linear invariant predictor and the maximum likelihood predictor for most of the cases considered. Further, we corrected the maximum likelihood predictor for bias and compared this modified predictor again with other predictors. Generally, it has found the modified maximum likelihood predictor is better than other predictors under consideration in the sense of PMC.

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CHAPTER ONE

Preliminaries

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1.1 Introduction

To predict an unobserved variable from one or more observed variables is frequent in statistical applications. In agriculture, for instance, one desires to know the grain yield for the next year. In fact, prediction is used in almost all fields of applications. For example, prediction is the main purpose of regression and time series analysis. Suppose $X_1, X_2, \ldots\ldots, X_n$ are an independent and identically distributed observed variables and Y is the variable to be predicted. Also, assume that $(X_1, X_2, \ldots\ldots, X_n, Y)$ have the joint distribution function $F(., \underset{\sim}{\theta})$, where $\underset{\sim}{\theta}$ is a vector of parameters belonging to a certain parameter space Ω. The prediction of Y is defined to be a function, $\hat{Y} = \hat{Y}(X_1, X_2, ..., X_n)$, of the observed variable $X_1, X_2, \ldots\ldots, X_n$. We know good estimation should satisfy some properties like unbiasedness, consistency, efficiency and sufficiency. As a matter of fact, the same is true for predictors. Unbiasedness, consistency and having small errors are basic requirements in a predictor. The predictor $\hat{Y}$ is said to be an unbiased predictor of Y if $E[\hat{Y} - Y] = 0$. On the other hand, $\hat{Y}$ is consistent if $\hat{Y}$ converges to Y in probability and converges in quadratic mean if $E[\hat{Y} - Y]^2$ converges to zero. Pitman (1937) introduced a comparative measure of closeness of two estimators of a parameter to its real value. If $\hat{\underset{\sim}{\theta}}_1$ and $\hat{\underset{\sim}{\theta}}_2$ are two estimators for $\underset{\sim}{\theta}$, then $\hat{\underset{\sim}{\theta}}_1$ is closer than $\hat{\underset{\sim}{\theta}}_2$ to $\underset{\sim}{\theta}$ if $\Pr(|\hat{\underset{\sim}{\theta}}_1 - \underset{\sim}{\theta}| < |\hat{\underset{\sim}{\theta}}_2 - \underset{\sim}{\theta}|) \geq \frac{1}{2}$. This concept has been utilized by many authors e.g. Balakrishnan, Davies, Keating and Mason (2011), and implemented to compare predictors. Variables sorted in ascending order are called ordered statistics, predicting future order statistics is a very common application in life testing data. The aim of this thesis is to compare predictors of future ordered statistics, of a sample from an exponential distribution with scale parameter σ based on the Pitman's Measure of Closeness

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(PMC). These predictors have been established and studied by many authors like Kaminsky and Rhodin (1985), Takada (1991), and Raqab (1992). Balakrishnan *et al* (2011) compare the best linear unbiased predictor (BLUP) and the best linear invariant predictor (BLIP) using the PMC. We will compare more predictors. Also, we will modify the maximum likelihood predictor and compare it to already known predictors based on the (PMC).

1.2 Order Statistics and their Corresponding Distributions

Order statistics play an important role in numerous practical applications such as the study of system reliability. A system of n components is called a k-out-of-n system, (see, Balakrishnan and Cohen, 1991, p. 3), if it remains operational only if at least k components continue to function. For components with independent lifetime distributions, the time to failure of the system is thus the $(n-k+1)^{th}$ order statistic. The special cases k=1 and k=n correspond respectively to parallel and series systems.

Suppose that $X_1, X_2, \ldots\ldots, X_n$ is a set of independent and identically distributed (*iid*) random variables with continuous cumulative distribution function (cdf) F(x) and probability density function (pdf) f(x). The corresponding order statistics are the X_i's arranged in nondecreasing order. The smallest of the X_i's is denoted by $X_{1:n}$, the second smallest value is denoted by $X_{2:n}$, . . ., and, finally, the largest value is denoted by $X_{n:n}$. Thus, $X_{1:n} \le X_{2:n} \le \ldots\ldots \le X_{n:n}$.

The pdf of the r^{th} order statistic $X_{r:n}$ (see, for example, Arnold et al., 1992) is given by

$$f_{r:n}(x) = c(r,n)\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r} f(x), \ -\infty < x < \infty. \tag{1.1}$$

Where $c(r,n) = \dfrac{n!}{(r-1)!(n-r)!}$.

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When r =1, the pdf of the minimum of a sample of size n, is given by

$$f_{1:n}(x) = n\left[1 - F(x)\right]^{n-1} f(x), \ -\infty < x < \infty.$$

For r = n, the pdf of the maximum of a sample of size n is

$$f_{n:n}(x) = n\left[F(x)\right]^{n-1} f(x), \ -\infty < x < \infty.$$

The joint pdf of any two order statistics $X_{r:n}$ and $X_{s:n}$, $(1 \le r < s \le n)$ is

$$f_{r,s:n}(x,y) = c(r,s,n)\left[F(x)\right]^{r-1}\left[F(y) - F(x)\right]^{s-r-1}[1 - F(y)]^{n-s} f(x) f(y),$$

$$-\infty < x < y < \infty. \quad (1.2)$$

Where $c(r,s,n) = \dfrac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

The cdf of the r^th order statistic $X_{r:n}$ is given by

$$F_{r:n} = \int_0^{F(x)} c(r,n)\, t^{r-1}(1-t)^{n-r} dt, \ -\infty<x<\infty. \quad (1.3)$$

Where $c(r,n)$ is given in (1.1).

The joint cdf of $X_{r:n}$ and $X_{s:n}$ is

$$F_{r,s:n}(x,y) = \int_0^{F(x)} \int_{t_1}^{F(y)} c(r,s,n)\, t_1^{r-1}(t_2 - t_1)^{s-r-1}(1-t_2)^{n-s} dt_2 dt_1, \ -\infty<x<y<\infty. \quad (1.4)$$

Where $c(r,s,n)$ is given in (1.2).

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1.3 Order Statistics from Exponential Distribution

Assume that $X_1, X_2, \ldots\ldots, X_n$ is a random sample from an exponential distribution with pdf

$$f(x;\mu,\sigma)=\frac{1}{\sigma}e^{-\frac{x-\mu}{\sigma}},\ x>\mu.$$

And cdf

$$F(x;\mu,\sigma)=1-e^{-\frac{x-\mu}{\sigma}},\ x>\mu.$$

If σ=1 and μ=0, the distribution is the standard exponential distribution. That is

$Z_i=\dfrac{X_i-\mu}{\sigma}$ distributed as a standard exponential distribution Exp(1). The order statistics corresponding to the standardized sample $Z_1, \ldots, Z_n$ are

$Z_{1:n}<Z_{2:n}<\ldots\ldots<Z_{n:n}$.

In what follows, we present some properties and results for order statistics from an exponential distribution. These results can be found, for example, in David and Nagaraja (2003).

Lemma 1.1: If $Z_{1:n}, Z_{2:n}, \ldots, Z_{n:n}$ are the order statistics from Exp (1), then

$$f_{r:n}(z)=c(r,n)(1-e^{-z})^{r-1}e^{-(n-r+1)};\ z>0. \tag{1.5}$$

Where $c(r,n)$ is given in (1.1).

Lemma 1.2: (Sukhatme 1937)

Let $Z_{0:n}\equiv 0$, and $Y_j=(n-j+1)(Z_{j:n}-Z_{j-1:n})$; j=1, 2,…., n, then $Y_1, Y_2, \ldots, Y_n$ are all independent and identically distributed variables with Exp(1) distribution.

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Lemma 1.3: Suppose $Z_{r:n}$ is the r[th] order statistic in a sample of size n from Exp(1) distribution, then (see Balakrishnan and Nagaraja 1992)

i) $\mu_{r:n} = E(Z_{r:n}) = \sum_{j=1}^{r} \frac{1}{n-j+1}$,

ii) $\sigma_{r,r:n} = Var(Z_{r:n}) = \sum_{j=1}^{r} \frac{1}{(n-j+1)^2}$,

iii) $\sigma_{r,s:n} = Cov(Z_{r:n}, Z_{s:n}) = \sum_{j=1}^{r} \frac{1}{(n-j+1)^2} = \sigma_{r,r:n}$; $r \le s$, and

iv) $\eta_{r:n} =$ Mode of $Z_{r:n} = \log(\frac{n}{n-r+1})$.

Proof: if Y_j is as defined in Lemma 1.2, then it is readily follows that

$$Z_{r:n} \stackrel{d}{=} \sum_{j=1}^{r} \frac{Y_j}{n-j+1}, \tag{1.6}$$

where $\stackrel{d}{=}$ means equality in distribution, and

i) $\mu_{r:n} = E(Z_{r:n}) = \sum_{j=1}^{r} \frac{E(Y_j)}{n-j+1} = \sum_{j=1}^{r} \frac{1}{n-j+1}$.

ii) $\sigma_{r,r:n} = Var(Z_{r:n}) = \sum_{j=1}^{r} \frac{Var(Y_j)}{(n-j+1)^2} = \sum_{j=1}^{r} \frac{1}{(n-j+1)^2}$.

iii) $\sigma_{r,s:n} = E(Z_{r:n} Z_{s:n}) - \mu_{r:n}\mu_{s:n}$

$$= E\left(\sum_{j=1}^{r} \frac{Y_j}{n-j+1} \sum_{k=1}^{s} \frac{Y_k}{n-k+1}\right) - \mu_{r:n}\mu_{s:n}$$

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$$=\sum_{j=1}^{r}\sum_{k=1}^{s}\frac{E(Y_jY_k)}{(n-j+1)(n-k+1)}-\mu_{r:n}\mu_{s:n}$$

$$=\sum_{j=1,j=k}^{r}\frac{2}{(n-j+1)^2}+\sum_{j=1}^{r}\sum_{\substack{k=1\\ j\neq k}}^{s}\frac{1}{(n-j+1)(n-k+1)}-\mu_{r:n}\mu_{s:n}\ ;$$

$$=\sum_{j=1,j=k}^{r}\frac{1}{(n-j+1)^2}+\sum_{j=1}^{r}\frac{1}{(n-j+1)}\sum_{k=1}^{s}\frac{1}{(n-k+1)}-\mu_{r:n}\mu_{s:n}$$

$$=\sum_{j=1}^{r}\frac{1}{(n-j+1)^2}.$$

iv) Differentiating (1.5) with respect to z and equating the result to zero, one obtains

$$\frac{df_{r:n}(z)}{dz}=c(r,n)(r-1)(1-e^{-z})^{r-2}e^{-z}e^{-(n-r+1)z}-c(r,n)(1-e^{-z})^{r-1}(n-r+1)\ e^{-(n-r+1)z}=0$$

Dividing by the common factors, the above is reduced to

$$(r-1)\ e^{-z}-(n-r+1)(1-e^{-z})=0$$

Upon arranging and simplifying, we have

$$z=\log\frac{n}{n-r+1}.$$

Now, we note that

$$\frac{df_{r:n}(z)}{dz}=c(r,n)(1-e^{-z})\ e^{-(n-r+1)z}\ (n\ e^{-z}-(n-r+1))>0 \text{ iff}$$

$$(n\ e^{-z}-(n-r+1))>0.$$

Therefore, $f_{r:n}(z)$ is increasing iff z < $\log\frac{n}{n-r+1}$ and decreasing iff

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z > $\log \frac{n}{n-r+1}$. Thus, $\eta_{r:n} = \log \frac{n}{n-r+1}$ is the mode of $Z_{r:n}$.

Lemma 1.4: The distribution function of the rth order statistic from Exp(1) is given by

$F_{r:n}(z) = 1 - I(r, n-r+1; e^{-z})$, where $I(\alpha, \beta; x)$ is the incomplete beta probability evaluated at x , and given by

$$I(\alpha, \beta; x) = \int_0^x \frac{u^{\alpha-1}(1-u)^{\beta-1}}{B(\alpha, \beta)} du.$$

Corollary 1.1:

The median of the rth order statistic $\tilde{\eta}_{r:n}$ from Exp(1) is the solution of the equation

$I(r, n-r+1; e^{-\tilde{\eta}_{r:n}})$ =0.5.

Lemma 1.5: Suppose $Z_{1:n}, \ldots, Z_{n:n}$ are the order statistic from Exp(1), and

$W = \sum_{j=1}^{r} (n-j+1)(Z_{j:n} - Z_{j-1:n})$; r=1, 2,….., n, then

W is distributed Gamma(r, 1), where Gamma(α, β) refers to a gamma distribution with shape parameter α and scale parameter β with density function is

$$f(w) = \frac{w^{r-1} e^{-w}}{\Gamma(r)}, w > 0. \tag{1.7}$$

1.4 Types of Censoring Data

By censored data (see, David and Nagaraja, 2003), we shall mean that, in a potential sample of size n, a known number of observations is missing at either end of the sample (single censoring) or at both ends (double censoring). In many life-testing

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experiments, it is quite common not to observe complete data, but only to observe some data set from censored data. This may be due to cost and/ or time limitations. For example: Fifty expensive machines are started up in an experiment. If, as is to be hoped, they were fairly reliable, it would take an enormously long time to wait for all machines to fail. Instead, great saving in time and machines usage can be affected if we base our estimates on the first few failure times, i.e., the first few order statistics from a sample of *(iid*) failure times.

Among types of censoring data, there is Type II censoring (see, Balakrishnan and Rao, 1998, p. 15). In this Type, the experimenter decides to observe only a pre-fixed number of failures, say r, and then stop the experiment as soon as the r^{th} failure occurs, thus censoring the last (n-r) unit still surviving. Thus, the number of failures is fixed, and the termination time is random.

1.5 Prediction of Order Statistics

As mentioned in the introduction prediction of future events (or that of events which have already occurred but were unobservable) on the basis of the knowledge about observed events is one of the fundamental problems of statistics. Consequently, several statistical estimation and inference procedures have been associated with the problem of predicting the behavior of one or more random variables that will be available in the future.

Suppose $X_{1:n}$, $X_{2:n}$, ..., $X_{n:n}$ are the observed r order statistics obtained from a sample of size n, from a location-scale family of distributions with density function

$$g(x;\mu,\sigma) = \frac{1}{\sigma} f(\frac{x-\mu}{\sigma}) .$$

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The random variable $Z = \dfrac{X - \mu}{\sigma}$ has standard distribution, free of the parameters μ and σ. In the context of prediction, prediction means expecting the value of an unobserved random variable ($X_{s:n}$, s > r) of interest, while estimation corresponds to expecting the value of an unknown, but fixed, parameter.

Prediction of future order statistics can be broadly classified under two categories (see, Nagaraja, 1995, p. 141):

(*i*) The variable to be predicted comes from the same experiment or sample, so that it may be correlated with the observed data. The problem in this category is called as one-sample prediction problem.

(*ii*) The variable to be predicted comes from an independent future experiment, and the problem in this category is two- or multi-sample prediction problem.

Suppose a machine consists of n components and fails whenever k, ($k \le n$) of these components fail. Such a machine is referred to as (n-k+1)-out-of-n system. Suppose our observations consist of the first r, ($r < k \le n$) failure times, and the goal is to predict the failure time of the machine. Assuming the components' life lengths are *(iid)*, then the prediction problem falls into Category (i). If we also suppose that the manufacturer of certain equipment is interested in setting up a warranty for the equipment in a lot being sent out to the market. Then by using the information based on a small sample, possibly censored, the goal is to predict and set a lower prediction limit for the weakest item in a future sample. Typical assumption here is that the two-samples are independent. This falls into Category (*ii*).

There are several prediction methods that have been introduced in the literature. The best linear unbiased predictor (BLUP) is the most popular predictor of censored

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order statistics. Kaminsky and Nelson (1975) applied the work of Goldberger (1962) to obtain the BLUP for Type-II right censored samples. The maximum likelihood predictor (MLP), which is based on maximizing the prediction likelihood function, was discussed by Kaminsky and Rhodin (1985).

Raqab (1992) has introduced the idea of conditional median predictors (CMP).It is known that the best unbiased predictor (BUP) of Y having observed $\boldsymbol{X}$ is $E_{\underset{\sim}{\theta}}\left[Y \mid \boldsymbol{X}\right]$ which, of course possibly depends on $\underset{\sim}{\theta}$. Instead of the conditional mean, one can use the conditional median and replace $\underset{\sim}{\theta}$ by its minimum variance unbiased estimator (MVUE) to produce a predictor of Y. Raqab calls this a CMP. Another type of predictors known as median unbiased predictor (MUP). A predictor $\hat{Y}$ of Y is a median unbiased predictor (MUP) if $P_{\underset{\sim}{\theta}}\left[\hat{Y} \leq Y\right] = P_{\underset{\sim}{\theta}}\left[\hat{Y} \geq Y\right]$. Takada (1991) has considered the property of median unbiasedness as invariant prediction problem. Kalbfleisch (1971) discussed general likelihood methods of prediction and Hinkley (1979) introduced a definition of predictive likelihood with close resemblance to Bayesian ideas. There are survey papers on prediction intervals by Hahn and Nelson (1973) and Patel (1989) and the book on statistical intervals by Hahn and Meeker (1991). David (1981) and Nelson (1982) included also a discussion on prediction problems. Raqab (1997) obtained modified maximum likelihood prediction of future order statistics from normal samples. Other prediction problems can be found in Lawless (1973), Kaminsky and Nelson (1975).

1.6 Pitman's Measure of Closeness

The comparisons of any statistic for example estimators based on different criteria have been received a great attention by researchers. For example, one may compare an unbiased estimator to a minimum mean squared error estimator. Rao

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(1981) recommended comparing estimators based on Pitman's measure of closeness. The Pitman closeness of $\hat{\underset{\sim}{\theta}}_1$ relative to $\hat{\underset{\sim}{\theta}}_2$ is the probability that an estimator $\hat{\underset{\sim}{\theta}}_1$ closer to real –valued parameter $\underset{\sim}{\theta}$ than the competitor $\hat{\underset{\sim}{\theta}}_2$, both estimators being based on a sample of size n. This paired measure of comparison was formally given by Pitman (1937) as following:

Definition1.1: Let $\hat{\underset{\sim}{\theta}}_1$ and $\hat{\underset{\sim}{\theta}}_2$ be two real–valued estimators of the real parameter $\underset{\sim}{\theta}$. Then Pitman's measure of closeness of these two competing estimators is denoted by $IP(\hat{\underset{\sim}{\theta}}_1, \hat{\underset{\sim}{\theta}}_2 \mid \underset{\sim}{\theta}) = \Pr(\left|\hat{\underset{\sim}{\theta}}_1 - \underset{\sim}{\theta}\right| < \left|\hat{\underset{\sim}{\theta}}_2 - \underset{\sim}{\theta}\right|)$.

Definition 1.2:(Pitman-Closer Estimators). Let $\hat{\underset{\sim}{\theta}}_1$ and $\hat{\underset{\sim}{\theta}}_2$ be two real –valued estimators of the real parameter $\underset{\sim}{\theta}$, then $\hat{\underset{\sim}{\theta}}_1$ is said to be a Pitman-closer estimate than $\hat{\underset{\sim}{\theta}}_2$ if $\Pr(\left|\hat{\underset{\sim}{\theta}}_1 - \underset{\sim}{\theta}\right| < \left|\hat{\underset{\sim}{\theta}}_2 - \underset{\sim}{\theta}\right|) \geq 0.5$, for all values of $\underset{\sim}{\theta}$, with strict inequality for at least one $\underset{\sim}{\theta}$.

Definition 1.3: (Pitman-Closest Estimator). Let D be a nonempty class of estimators of a common parameter $\underset{\sim}{\theta}$. Then $\hat{\underset{\sim}{\theta}}^*$ is Pitman-closest among the estimators in D provided for every $\hat{\underset{\sim}{\theta}} \in$ D, such that $\hat{\underset{\sim}{\theta}}^* \neq \hat{\underset{\sim}{\theta}}$, Pr $(\left|\hat{\underset{\sim}{\theta}}^* - \underset{\sim}{\theta}\right| < \left|\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta}\right|) \geq 0.5$, for all values of $\underset{\sim}{\theta}$, with strict inequality for at least on value of $\underset{\sim}{\theta}$.

Whenever both estimators are functions of a common statistic, (as often occurs in the presence of sufficient statistic), we will denote $IP(\hat{\underset{\sim}{\theta}}_1, \hat{\underset{\sim}{\theta}}_2 \mid \underset{\sim}{\theta})$ as π_n. This pairwise comparison has been extended by Pitman to include all estimators in a class D. An estimator $\hat{\underset{\sim}{\theta}}_j$ is Pitman closeness inadmissible within a class D if there is an estimator

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$\hat{\underset{\sim}{\theta}}_i$ in D which is Pitman-closer than $\hat{\underset{\sim}{\theta}}_j$. For a comprehensive review of the Pitman's measure of closeness as a comparison criterion of estimators, one may refer to Keating *et al* (1993). By considering a two-parameter exponential distribution, Nagaraja (1986) compared the best linear unbiased estimators (BLUE) and the best linear invariant estimators (BLIE) of the location and scale parameters based on Type-II censored sample.

In this thesis, continuing along the lines of Nagaraja (1986), by assuming a Type-II right censored sample from a scaled exponential distribution, we compare some predictors of censored order statistics in the one-sample case using Pitman's measure of closeness. Other type of predictors, including MLP and CMP, will also be considered.

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CHAPTER TWO

Predictors of Future Order Statistics from Exponential Distribution

1.1 Maximum Likelihood Predictor.

1.2 The Best Linear Unbiased Predictor.

1.3 The Best Linear Invariant Predictor.

1.4 Conditional Median Predictor.

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In this chapter, we review and reproduce different types of predictors that have been proposed in literature to predict future order statistics based on an observed type II censored sample from an exponential distribution. The predictors that will be considered are the Maximum Likelihood Predictors (MLP), Best Linear Unbiased Predictor (BLUP), Best Linear Invariant Predictor (BLIP) and Conditional Median Predictor (CMP).

Throughout this chapter, we will assume that $X_{1:n}$, $X_{2:n}$, ..., $X_{n:n}$ are the order statistics from Exp(σ) and that $Z_{1:n}$, $Z_{2:n}$, ..., $Z_{n:n}$, $Z_{i:n} = \dfrac{X_{i:n}}{\sigma}$, i=1,2,…, n, are the order statistics from Exp(1). Further we assume that $X_{1:n}$, $X_{2:n}$, ..., $X_{r:n}$ are observed, and $X_{s:n}$; s>r, is to be predicted.

2.1 Maximum Likelihood Predictor (MLP)

Kaminsky and Rhodin (1985) applied the principle of maximum likelihood to the joint estimation and prediction of a future order statistic and an unknown parameter in the one-sample prediction problem. The goal is to predict higher order statistics from the lower ones in Type II censored random sample. The procedure used here is the classical maximum likelihood method of estimation that explained below.

Definition 2.1: let X_1, X_2, ..., X_n (iid) with pdf $f\,(X\,,\ \underset{\sim}{\theta}), \underset{\sim}{\theta} \in \Omega$, the MLE of $\underset{\sim}{\theta}$ is the value that maximizes the Likelihood function

$$\mathrm{L}(\underset{\sim}{\theta}) = f\,(X_1,\, X_2,\, ...,\, X_n,\ \underset{\sim}{\theta}) = \prod_{i=1}^{n} f\,(X_i\,,\ \underset{\sim}{\theta})$$

(see, for example, Hogg and Craig, 1978, p. 202) .

Let $\underline{X} = (X_1,\ X_2,\ ...,\ X_r)$ and $\underline{Y} = (Y_1,\ Y_2,\ ...,\ Y_p)$ denote random vectors with joint pdf f($\underline{x}$, $\underline{y}$; $\underset{\sim}{\theta}$) indexed by the parameter $\underset{\sim}{\theta} \in \Omega$. If $\underline{X}$ and $\underline{Y}$ could both be observed, f (.) would correspond to usual likelihood function of $\underset{\sim}{\theta}$. The problem here

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is to predict the unobserved (either future, or past and missing) value of $\underline{Y}$, having observed $\underline{X}$. The dependency between $\underline{X}$ and $\underline{Y}$ is assumed (although their components need not be). Thus, the predictive likelihood function (PLF) can be defined as $L(\underline{y},\ \underset{\sim}{\theta};\ \underline{x}) = \mathrm{f}\left(\underline{x},\ \underline{y};\ \underset{\sim}{\theta}\right)$.

Let $\hat{\underline{\mathrm{Y}}} = \mathrm{t}(\underline{\mathrm{x}})$, and $\hat{\underset{\sim}{\theta}} = \mathrm{u}(\underline{\mathrm{x}})$ be statistics, of which $L(\hat{\underline{Y}},\ \hat{\underset{\sim}{\theta}};\ \underline{X}) = \sup\limits_{(\underline{y},\theta)} L(\underline{y},\ \underset{\sim}{\theta};\ \underline{x})$.

The statistic $\hat{\underline{\mathrm{Y}}} = \mathrm{t}(\underline{\mathrm{x}})$ is called the MLP of $\underline{Y}$, and $\hat{\underset{\sim}{\theta}} = \mathrm{u}(\underline{\mathrm{x}})$ is called the predictive maximum likelihood estimator (PMLE) of $\underset{\sim}{\theta}$.

Now, let $\underline{X} = (X_{1:n}, X_{2:n}, \ldots, X_{r:n})$, $1 \le \mathrm{r} \le \mathrm{n}$, where $X_{i:n}$ $1 \le \mathrm{i} \le \mathrm{r}$ is the i^{th} failure time. Then the PLF of $X_{s:n}$ ($\mathrm{r} < \mathrm{s} \le \mathrm{n}$), and $\underset{\sim}{\theta}$ given $\underline{X}$ is

$$\mathrm{L}\left(\mathrm{x}_{\mathrm{s:n}}, \underset{\sim}{\theta}; \mathrm{X}\right)$$

$$= c_n(r,s)\prod_{j=1}^{r} f(x_{j:n})\left[F(x_{s:n}) - F(x_{r:n})\right]^{s-r-1}\left[1 - F(x_{s:n})\right]^{n-s} f(x_{s:n}),$$

$$-\infty < x_{1:n} < x_{2:n} < \ldots < x_{s:n} < \infty \tag{2.1}$$

Where $\mathrm{c_n(r,s)} = \dfrac{\mathrm{n}!}{(\mathrm{s}-\mathrm{r}-1)!(\mathrm{n}-\mathrm{s})!}$

The log likelihood function can be written as

$$\log L \ \propto \ \sum_{j=1}^{r} f(x_{j:n}) + (s-r-1)\ \log\ \left[F(x_{s:n}) - F(x_{r:n})\right] + \ (n-s)\ \log\ \left[1 - F(x_{s:n})\right] + \log f(x_{s:n}).$$

Now, if there exists a unique solution, $\hat{X}_{s:n}$, of the predictive likelihood equation (PLE)

$\dfrac{\partial \log L}{\partial x_{s:n}} = 0$, then $\hat{X}_{s:n}$ must be the unique MLP of $X_{s:n}$. The PLE is given by

$$\frac{\partial \log L}{\partial x_{s:n}} = \left[(s-r-1)\frac{f(x_{s:n})}{F(x_{\mathrm{s:n}}) - F(x_{\mathrm{r:n}})} - (n-s)\frac{f(x_{s:n})}{1 - F(x_{\mathrm{s:n}})} + \frac{f'(x_{\mathrm{s:n}})}{f(x_{\mathrm{s:n}})}\right] = 0.$$

Simplifying the above equation, we have

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$$\frac{\partial log\ L}{\partial x_{s:n}} = \left[\frac{f'(x_{\text{s:n}})}{f^2(x_{\text{s:n}})} + \frac{(s-r-1)}{F(x_{\text{s:n}})-F(x_{\text{r:n}})} - \frac{n-s}{1-F(x_{\text{s:n}})}\right] = 0. \tag{2.2}$$

Notice that from (2.2) when $\underset{\sim}{\theta}$ is known, the MLP of $X_{s:n}$, if it exists, is a function of $X_{r:n}$ and the known value of $\underset{\sim}{\theta}$.

Kaminsky and Rhodin (1985) derived the MLP and the PMLE for $X_{s:n}$ and σ, respectively, when the observed censored data is sampled from an exponential distribution with scale parameter σ, $\sigma > 0$.In what follows, we give a detailed derivation of their results.

Lemma 2.1: Let $X_{1:n}$, ..., $X_{r:n}$ be an observed ordered sample from Exp(σ), then the MLP of $X_{s:n}$, $r \le s \le n$ is $\delta_{MLP} = X_{r:n} + \sigma\pi_0$, when σ is known, and

$\delta_{MLP} = X_{r:n} + \dfrac{\pi_0}{r+1}T$, when σ is unknown. Where $\pi_0 = \log(\dfrac{n-r}{n-s+1})$ is the mode of

the order statistic $Z_{s-r:n-r}$ sampled from Exp(1), and $T = \sum_{i=1}^{r} X_{i:n} + (n-r)X_{r:n}$.

Proof: By (2.1), the PLF of $X_{s:n}$ and σ can be written as

$$L\left(X_{s:n},\sigma;X\right) = c_n(r,s)\ \prod_{\text{j=1}}^{\text{r}}\ (\sigma^{-1}.\ e^{-\frac{X_{j:n}}{\sigma}})(e^{-\frac{X_{r:n}}{\sigma}} - e^{-\frac{X_{s:n}}{\sigma}})^{s-r-1}(e^{-\frac{X_{s:n}}{\sigma}})^{\text{n-s}}\ (\sigma^{-1}.\ e^{-\frac{X_{s:n}}{\sigma}}).$$

Taking the logarithm of both sides of the above terms, one has

$$\log L(X_{s:n},\sigma,X) \propto -(r+1)\log\sigma - \frac{\sum_{j=1}^{r} X_{j:n} + (n-s+1)\ X_{s:n}}{\sigma}$$

$$+\ (s-r-1)\ \log\left[e^{-\frac{X_{r:n}}{\sigma}} - e^{-\frac{X_{s:n}}{\sigma}}\right]. \tag{2.3}$$

Differentiating (2.3) with respect to $X_{s:n}$ and equating to zero, we obtain

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$$\frac{\partial \log L}{\partial x_{s:n}} = -\frac{(n-s+1)}{\sigma} + (s-r-1)\frac{\frac{1}{\sigma}e^{-\frac{X_{s:n}}{\sigma}}}{\left[e^{-\frac{X_{r:n}}{\sigma}} - e^{-\frac{X_{s:n}}{\sigma}}\right]} = 0,$$

which implies that $X_{s:n} = X_{r:n} + \sigma \log(\frac{n-r}{n-s+1})$ (2.4)

In fact the right hand side of (2.4) is the MLP of $X_{s:n}$ if σ is assumed to be known; that is $\hat{X}_{s:n} = X_{r:n} + \sigma\pi_0$. By the way, as shown in Lemma 1.4 (iv), π_0 is the mode of the order statistic $Z_{s-r:n-r}$ sampled from Exp(1).

Replacing $X_{s:n}$ by $X_{r:n} + \sigma\pi_0$, then differentiating (2.3) with respect to σ and equating to zero, we obtain

$$\frac{\partial \log L}{\partial \sigma} = -\frac{(r+1)}{\sigma} + \frac{\sum_{j=1}^{r} X_{j:n} + (n-s+1)\, X_{s:n}}{\sigma^2} + \frac{(s-r-1)}{\sigma^2} X_{r:n} = 0 \qquad (2.5)$$

Upon simplifying and rearranging, the above yield the PLME as

$$\hat{\sigma} = \frac{\sum_{j=1}^{r} X_{j:n} + (n-r) X_{r:n}}{r+1} = \frac{T}{r+1},$$

where T= $\sum_{j=1}^{r} X_{j:n} + (n-r) X_{r:n}$, (2.6)

and the MLP of $X_{s:n}$, If σ is unknown is

$$\hat{X}_{s:n} = X_{r:n} + \frac{T}{r+1}\pi_0. \qquad (2.7)$$

The mean square prediction error (MSPE), when σ unknown, of this predictor is

$$\text{MSPE } (\delta_{MLP}) = E(X_{s:n} - \hat{X}_{s:n})^2$$

$$= \text{Var}(X_{s:n} - \hat{X}_{s:n}) + \left(E(X_{s:n} - \hat{X}_{s:n})\right)^2$$

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$$= \text{Var}(X_{s:n} - X_{r:n} - \pi_0 \frac{T}{r+1}) + \left(E(X_{s:n} - X_{s:n} - \pi_0 \frac{T}{r+1}) \right)^2$$

We have , $X_{s:n} - X_{r:n}$ and $X_{r:n}$ are independent which implies that $X_{s:n} - X_{r:n}$ and T are independent . Thus, the right hand side term above reduces to:

$$\text{MSPE}(\delta_{MLP}) = \text{Var}(X_{s:n}) - \text{Var}(X_{r:n}) + \pi^2{}_0 \ \text{Var}(\frac{T}{r+1}) + \left(E(X_{s:n} - X_{s:n} - \pi_0 E(\frac{T}{r+1}) \right)^2$$

$$= \sigma^2 \left(\left(\pi_2 + \pi_1^2 + \pi_0 (\frac{r}{r+1})(\pi_0 - 2\pi_1) \right) \right),$$ where π_1 is the mean and π_2 is the variance of the order statistic $Z_{s-r:n-r}$ sampled from Exp(1).

2.2 The Best Linear Unbiased Predictor (BLUP)

The best linear unbiased predictor (BLUP) of the regression parameters in the generalized linear model was derived by Goldberger (1962). Kiminsky and Nelson (1975) derived the BLUP of the future observation $X_{s:n}$ based on the observed values $X_{1:n}, \ldots, X_{r:n}, 1 \le r \le s \le n$, where $X_{1:n}, \ldots, X_{r:n}$ are the order statistics of a random sample of size n from a continuous location-scale family pdf $f(x) = \frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$.

Let $\underset{\sim}{X}' = (X_{1:n}, \ldots, X_{r:n})$, $\underset{\sim}{\alpha}' = (\alpha_1, \ldots, \alpha_r)$, where $\alpha_i = E(\frac{X_{i:n} - \mu}{\sigma}) = E(Z_{i:n})$. Further denote the variance covariance matrix of $\underset{\sim}{X}$ by $\sigma^2 V$, and let $\underset{\sim}{W}' = (W_1, \ldots, W_r)$ denote the covariance vector between $Z_{i:n}$ and $Z_{s:n}$ for $1 \le i \le r$. Kaminsky and Nelson (1975) derived the BLUP for $X_{s:n}$ as

$\delta_{BLUP} = \hat{X}_{s:n} = (\hat{\mu} + \hat{\sigma}\alpha_s) + W' V^{-1}(\underset{\sim}{X} - \hat{\mu}1 - \hat{\sigma}\underset{\sim}{\alpha})$, where $\underset{\sim}{1}' = (1,1,\ldots..,1)$, $\hat{\mu}$ and $\hat{\sigma}$ are the BLUE of μ and σ respectively.

For the exponential distribution with scale parameter σ, it can be shown that

$W' V^{-1} = (0,0\ldots\ldots..,1)$, (Balakrishnan and Rao p 437). Thus, $\delta_{BLUP} = X_{r:n} + (\alpha_s - \alpha_r)\hat{\sigma}$.

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Applying the BLUE given by Lloyed (1952), we obtain

$$\hat{\sigma} = \frac{\sum_{i=1}^{r} X_{i:n} + (n-r)X_{r:n}}{r} = \frac{T}{r}.$$

Therefore, $\delta_{BLUP} = X_{r:n} + \frac{\pi_1}{r}T$. (2.8)

And the (MSPE) of this predictor is

$$E(X_{s:n} - \hat{X}_{s:n})^2 = \text{Var}(X_{s:n} - \hat{X}_{s:n}) + \left(E(X_{s:n} - \hat{X}_{s:n})\right)^2$$

$$= \text{Var}(X_{s:n} - X_{r:n} - \pi_1 \frac{T}{r}) + \left(E(X_{s:n} - X_{s:n} - \pi_1 \frac{T}{r}) \right)^2$$

$$= \text{Var}(X_{s:n}) - \text{Var}(X_{r:n}) + \pi^2{}_1 \text{Var}(\frac{T}{r}) + \left(E(X_{s:n} - X_{s:n} - \pi_1 E(\frac{T}{r}) \right)^2$$

$$= \sigma^2 \left(\pi_2 + \frac{\pi_1^2}{r} \right).$$

It is important to note that if σ is known, then $\delta_{BLUP} = X_{r:n} + \sigma\pi_1$.

2.3 The Best Linear Invariant Predictor (BLIP)

In this section, we present another predictor related to the generalized linear model, called the best linear invariant predictor. By (Takada 1991) we say that a statistic δ ($\underset{\sim}{X}$) is an invariant predictor of $X_{s:n}$ if for any (a, b) with a > 0,

$\delta(a\underset{\sim}{X} + b\underset{\sim}{1}) = a\delta(\underset{\sim}{X}) + b$, where $\underset{\sim}{1}' = (1,1,....1)$. In fact $\delta(\underset{\sim}{X})$ is the best linear invariant predictor (BLIP) for $X_{s:n}$ if $E(\delta(\underset{\sim}{X}) - X_{s:n})^2$ is minimum and proportional to σ^2.

Takada (1981) obtained a necessary and sufficient condition for a predictor to be best invariant predictor in location and scale families. He proved that the best invariant predictor can be expressed as a linear combination of the best unbiased predictor and the best unbiased estimator of the scale parameter. Actually, he shown that

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$$\delta_{BLIP} = \delta_{BLUP} + (c1/c2)\hat{\sigma}, \tag{2.9}$$

where δ_{BLUP} is the best linear unbiased predictor of $X_{s:n}$, δ_{BLIP} is the best linear invariant predictor of $X_{s:n}$,

$c1 = E\{(X_{s:n} - \delta_{BLUP})\hat{\sigma}\} = -Cov\{(1 - W'\,V^{-1}1)\hat{\mu} + (\alpha_s - W'\,V^{-1}\alpha)\hat{\sigma}, \hat{\sigma})\}$, and

$c2 = E(\hat{\sigma}^2)$.

For the exponential distribution with scale parameter σ, Takada (1991) showed that

$c1 = \dfrac{-\pi_1}{r}$, and $c2 = \dfrac{r+1}{r}$.

Thus,

$$\delta_{BLIP} = X_{r:n} + \frac{T}{r+1}\pi_1, \tag{2.10}$$

The (MSPE) of this predictor is

$$E(X_{s:n} - \hat{X}_{s:n})^2 = \mathrm{Var}(X_{s:n} - \hat{X}_{s:n}) + \left(E(X_{s:n} - \hat{X}_{s:n})\right)^2$$

$$= \mathrm{Var}(X_{s:n} - X_{r:n} - \pi_1 \frac{T}{r+1}) + \left(E(X_{s:n} - X_{s:n} - \pi_1 \frac{T}{r+1})\right)^2$$

$$= \mathrm{Var}(X_{s:n}) - \mathrm{Var}(X_{r:n}) + \pi_1^2\ \mathrm{Var}(\frac{T}{r+1}) + \left(E(X_{s:n} - X_{s:n} - \pi_1 E(\frac{T}{r+1})\right)^2$$

$$= \sigma^2\left(\pi_2^2 + \frac{\pi_1^2}{r+1}\right).$$

2.4 The Conditional Median Predictor (CMP)

This predictor depends on the median of conditional variate $X_{s:n}|X_{r:n}$. The conditional pdf of $X_{s:n}|X_{r:n}$ is given by

$$f_{X_{s:n}|X_{r:n}}(y|x) = \frac{f_{X_{r:n},X_{s:n}}(x,y)}{f_{X_{r:n}}(x)}$$

$$= \frac{c(r,s,n)(1-e^{-x/\sigma})^{r-1}(e^{-x/\sigma} - e^{-y/\sigma})^{s-r-1}\,\frac{1}{\sigma}(e^{-y/\sigma})^{n-s+1}\,\frac{1}{\sigma}e^{-y/\sigma}}{c(r,n)(1-e^{-x/\sigma})^{r-1}(e^{-x/\sigma})^{n-r}\,\frac{1}{\sigma}e^{-x/\sigma}}$$

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$$=\frac{c(r,s,n)(e^{-x/\sigma}-e^{-y/\sigma})^{s-r-1}\frac{1}{\sigma}(e^{-y/\sigma})^{n-s+1}e^{-y/\sigma}}{c(r,n)(e^{-x/\sigma})^{n-r}\frac{1}{\sigma}e^{-x/\sigma}}$$

$$=b_{r,s,n}\ (1-e^{-(y-x)/\sigma})^{s-r-1}\frac{1}{\sigma}e^{-(n-s+1)(y-x)/\sigma},\ \text{y > x.}$$

Where $b_{r,s,n}=\frac{(n-r)!}{(s-r-1)!(n-s)!}$, $c(r,s,n)$ and $c(r,n)$ are given in (1.2) and (1.1), respectively.

Thus, the cdf of $X_{s:n}|X_{r:n}$ is:

$$F_{X_{s:n}|X_{r:n}}(y|x)=\int_{x_r}^{y}f_{X_{s:n}|X_{r:n}}(x_{s:n}|x_{r:n})\,dx_{s:n}\,.$$

The median of $X_{s:n}$ given $X_{r:n}$ is y_0 satisfying

$$F_{X_{s:n}|X_{r:n}}(y_0)=\frac{1}{2},\ \text{or}\ b_{r,s,n}\int_{x}^{y_0}(1-e^{-(y-x)/\sigma})^{s-r-1}\frac{1}{\sigma}e^{-(n-s+1)(y-x)/\sigma}dy=\frac{1}{2}\,.$$

Putting $t=\frac{y-x}{\sigma}$, leads to

$$\frac{1}{2}=b_{r,s,n}\int_{0}^{t_0}(1-e^{-t})^{s-r-1}e^{-(n-s+1)t}e^{-t}dt\ ,\ \text{where}\ t_0=\frac{y_0-x}{\sigma}\,.$$

Hence, by Lemma 1.4 $F^{*}_{s-r:n-r}(t_0)=\frac{1}{2}$, where $F^{*}_{s-r:n-r}$ is the cdf of $Z_{s-r:n-r}=\frac{X_{s-r:n-r}}{\sigma}$.

Therefore, the CMP of $X_{s:n}$ is given by

$$\delta_{CMP}=X_{r:n}+\sigma Med(Z_{s-r:n-r})\,. \tag{2.11}$$

For unknown σ, it known that the UMVUE of σ is $\hat{\sigma}=\frac{\sum_{i=1}^{r}X_{i:n}+(n-r)X_{r:n}}{r}=\frac{T}{r}$.

Thus, the CMP of $X_{r:n}$ becomes

$$\delta_{CMP}=X_{r:n}+\frac{T}{r}Med(Z_{s-r:n-r})\,. \tag{2.12}$$

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CHAPTER THREE

The Comparisons between the Predictors under Pitman's Measure of Closeness

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In this chapter, we compare the BLUP, BLIP, MLP, and CMP predictor in terms of Pitman Measure of Closeness (PMC). These comparisons are based on a type II censored sample from an exponential distribution. Both cases, when the scale parameter σ is known and unknown are treated. Further, tables illustrating computations of the Pitman closeness measure probabilities are displayed. Let $X^{*}_{s:n}$ and $X^{**}_{s:n}$ be any two predictors of $X_{s:n}$, then we denote $\pi_{r,s,n} = \Pr\left(\left|X_{s:n} - X^{*}_{s:n}\right| < \left|X_{s:n} - X^{**}_{s:n}\right|\right)$.

3.1 BLUP vs. BLIP

Balakrishnan, Davies, Keating and Mason (2011) used Pitman's measure of closeness to compare Best Linear Unbiased and the Best Linear Invariant Predictor for $X_{s:n}$. As we will follow their derivations in the other comparisons, we reproduce their work and, more or less, adopt their notations. BLUP and BLIP of $X_{s:n}$, s > r, based on a Type-II right censored sample $X_{1:n} \leq \ldots\ldots \leq X_{r:n}$ from Exp(σ) distribution, which are the same when σ known, are given by

$\delta_{BLUP} = \delta_{BLIP} = X_{r:n} + \sigma\pi_1$.

Now, for unknown σ, we have

$$\delta_{BLUP} = X_{r:n} + \frac{T}{r}\pi_1, \text{ and } \delta_{BLIP} = X_{r:n} + \frac{T}{r+1}\pi_1, \tag{3.1}$$

where $T = \sum_{i=1}^{r} X_{i:n} + (n-r)\, X_{r:n}$.

Let $\pi_{r,s:n} = \Pr(\left|X_{s:n} - \delta_{BLUP}\right| < \left|X_{s:n} - \delta_{BLIP}\right|)$, then the following results are needed to compute $\pi_{r,s:n}$.

Lemma 3.1: If $Z_{1:n}$, $Z_{2:n}$, ..., $Z_{r:n}$ are the order statistics from Exp(1), then

$$f_{Z_{r:n}}(z) = c(r,n)\ (1-e^{-z})^{r-1}\ e^{-(n-r+1)z}, \tag{3.2}$$

where $c(r,n)$ is given in (1.1).

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Proof: This follows directly from (1.1)

Lemma 3.2: $f_{Z_{s:n}|Z_{r:n}}(v\,|u)=\dfrac{(n-r)!}{(s-r-1)!(n-s)!}(e^{-u}-e^{-v})^{s-r-1}e^{-(n-s+1)v}e^{(n-r)u}$, v > u > 0.

Proof: Follows from (3.2) and (1.2).

Lemma 3.3: If $U=Z_{s:n}-Z_{r:n}$, then

$f_U(u)=b_{r,s,n}(1-e^{-u})^{s-r-1}e^{-(n-s+1)u}$, u > 0, where $b_{r,s,n}=\dfrac{(n-r)!}{(s-r-1)!(n-s)!}$.

Proof: $f_U(u)=\int_0^\infty f_{Z_s-Z_r|Z_r=z}(u)f_{Z_r}(z)\,dz$

$$=\int_0^\infty f_{Z_s|Z_r}(u+z)f_{Z_r}(z)\,dz.$$

Now, by substitute the value of $f_{Z_r}(z)$ in Lemma (3.1) and $f_{Z_s|Z_r}(u+z)$ in Lemma (3.2), to get

$$f_U(u)=b_{r,s,n}c(r,n)\int_0^\infty (e^{-z}-e^{-(u+z)})^{s-r-1}e^{-(n-s+1)(u+z)}e^{(n-r)z}(1-e^{-z})^{r-1}e^{-(n-r+1)z}dz$$

$$=b_{r,s,n}c(r,n)\int_0^\infty (1-e^{-u})^{s-r-1}e^{-(n-s+1)u}(1-e^{-z})^{r-1}e^{-(n-r+1)z}dz$$

$$=b_{r,s,n}c(r,n)(1-e^{-u})^{s-r-1}e^{-(n-s+1)u}\int_0^\infty (1-e^{-z})^{r-1}e^{-(n-r+1)z}dz\ . \tag{3.3}$$

Putting y $=1-e^{-z}$ in (3.3), the integral is then equal to B(r, n-r+1). Hence,

$f_U(u)=b_{r,s,n}$ c(r, n) B(r, n-r+1) $(1-e^{-u})^{s-r-1}e^{-(n-s+1)u}$

$=b_{r,s,n}(1-e^{-u})^{s-r-1}e^{-(n-s+1)u}$, u > 0, and the result is obtained.

Now, we prove the following theorem by Balakrishnan, Davies, Keating and Mason (2011).

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Theorem 3.1: Let $X_{1:n}$, $X_{2:n}$, ..., $X_{r:n}$ be a sample from Exp(σ), then the desired Pitman closeness probability of BLUP and BLIP as predictor of $X_{s:n}$ is

$$\pi_{r,s:n} = b_{r,s,n} \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \frac{1}{(n-s+i+1)} \{(n-s+i+1)D+1\}^{-r},$$

where $D = \frac{\pi_1}{2}(\frac{1}{r} + \frac{1}{r+1})$.

Proof: We have $\pi_{r,s:n} = \Pr(|X_{s:n} - \delta_{BLUP}| < |X_{s:n} - \delta_{BLIP}|)$

$$= \Pr\left(\left|X_{s:n} - X_{r:n} - \frac{T}{r}\pi_1\right| < \left|X_{s:n} - X_{r:n} - \frac{T}{r+1}\pi_1\right|\right).$$

Dividing by σ, we get

$$\Pr\left(\left|U - \frac{\pi_1}{r} W\right| < \left|U - \frac{\pi_1}{r+1} W\right|\right), \tag{3.4}$$

where $U = Z_{s:n} - Z_{r:n}$ and $W = \frac{T}{\sigma} = \sum_{i=1}^{r} Z_{i:n} + (n-r)Z_{r:n}$.

We have $W \sim Gamma(r,1)$, (see, Blakrishnan and Nagaraja 1992)

Squaring both sides of the inequality in (3.4), one has

$$\Pr\left((U - \frac{\pi_1}{r} W)^2 < (U - \frac{\pi_1}{r+1} W)^2\right),$$

which is, upon simplification, reduced to

$$\Pr\left(2(\frac{\pi_1}{r+1} - \frac{\pi_1}{r})\, W\, U > (\frac{\pi_1^2}{(r+1)^2} - \frac{\pi_1^2}{r^2})\, W^2\right).$$

Since W is positive with probability one, the above reduces to:

$\Pr(U > D\, W)$, where $D = \frac{\pi_1}{2}(\frac{2r+1}{r(r+1)})$.

By Lemma 3.3, we have

$$\Pr(U > D\, W) = \int_0^{\infty} \Pr(U > D\, W \mid W)\, f_W(w)\, dw$$

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$$=b_{r,s,n}\int_0^\infty \Big(\int_{Dw}^\infty (1-e^{-u})^{s-r-1} e^{-(n-s+1)\,u} du\Big)\frac{1}{\Gamma(r)} w^{\,r-1} e^{-w}\, dw$$

$$=b_{r,s,n}\sum_{i=0}^{s-r-1}(-1)^i\binom{s-r-1}{i}\int_0^\infty\int_{DW}^\infty e^{-(n-s+i+1)\,u}\,\frac{1}{\Gamma(r)} w^{\,r-1}e^{-w}du\;dw$$

$$=b_{r,s,n}\sum_{i=0}^{s-r-1}(-1)^i\binom{s-r-1}{i}\frac{1}{(n-s+i+1)\Gamma(r)}\int_0^\infty w^{\,r-1}e^{-\{(n-s+i+1)D+1\}\,w}dw$$

$$=b_{r,s,n}\sum_{i=0}^{s-r-1}(-1)^i\binom{s-r-1}{i}\frac{1}{(n-s+i+1)}\{(n-s+i+1)D+1\}^{-r}. \qquad (3.5)$$

The values of Pitman closeness probabilities, when σ is unknown, for n=10 and 15 are presented in Tables 1 and 2, respectively .The results reveal that the BLUP is Pitman-closer than the BLIP when r=1, the BLIP is always Pitman-closer when s=r+1 except for r=1. In general, for small r, the BLUP is Pitman-closer while the BLIP is Pitman-closer for large values of r.

Table 1: Pitman Closeness Probability of BLUP verses BLIP, σ is unknown, n=10.

r\s	2	3	4	5	6	7	8	9	10
1	0.57	0.63	0.66	0.68	0.69	0.69	0.70	0.70	0.69
2		0.49	0.56	0.59	0.61	0.62	0.63	0.63	0.63
3			0.46	0.52	0.56	0.57	0.58	0.59	0.58
4				0.44	0.50	0.53	0.55	0.56	0.55
5					0.43	0.49	0.51	0.53	0.53
6						0.42	0.47	0.50	0.50
7							0.41	0.46	0.48
8								0.40	0.45
9									0.40

Table2: Pitman Closeness Probability of BLUP verses BLIP, σ is unknown, n=15.

r\s	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0.57	0.63	0.66	0.68	0.69	0.69	0.70	0.70	0.70	0.71	0.71	0.71	0.71	0.70
2		0.49	0.56	0.59	0.61	0.62	0.63	0.64	0.64	0.65	0.65	0.65	0.65	0.64
3			0.46	0.53	0.56	0.58	0.59	0.60	0.60	0.61	0.61	0.61	0.61	0.60
4				0.44	0.50	0.53	0.55	0.56	0.57	0.58	0.58	0.59	0.58	0.57
5					0.43	0.49	0.52	0.53	0.55	0.55	0.56	0.56	0.56	0.55
6						0.42	0.47	0.50	0.52	0.53	0.54	0.54	0.55	0.53
7							0.41	0.47	0.49	0.51	0.52	0.53	0.53	0.52
8								0.40	0.46	0.48	0.50	0.51	0.51	0.50
9									0.40	0.45	0.48	0.49	0.50	0.49
10										0.40	0.45	0.47	0.48	0.48
11											0.39	0.44	0.46	0.47
12												0.39	0.44	0.45
13													0.39	0.43
14														0.39

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We will complete the comparisons between the remaining predictors under (PMC) as same manner, and we will table the PMC probabilities. It is important to mention that for exponential distribution $\pi_0 < m < \pi_1$. Where π_0, π_1, and m are the mode, mean and median of the order statistic $Z_{s-r:n-r}$ sampled from Exp(1), respectively.

3.2 MLP vs. BLUP

Theorem 3.2: Let $X_{1:n}$, $X_{2:n}$, ..., $X_{r:n}$ be a sample from Exp(σ), then the desired Pitman closeness probability of MLP and BLUP as predictors of $X_{s:n}$, when σ is known, is

$$\pi_{r,s:n} = 1 - \frac{(n-r)!}{(s-r-1)!(n-s)!} \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \frac{e^{-D(n-s+i+1)}}{n-s+i+1},$$

where $D = \dfrac{\pi_1 + \pi_0}{2}$.

Proof: We have that the BLUP and MLP of $X_{s:n}$, when σ is known, are given by

$\delta_{BLUP} = X_{r:n} + \sigma\pi_1, \delta_{MLP} = X_{r:n} + \sigma\pi_0$, respectively .

Then, the desired Pitman closeness probability becomes

$\pi_{r,s:n}$=Pr $(|X_{s:n} - \delta_{MLP}| < |X_{s:n} - \delta_{BLUP}|)$.

Upon using the expression of the BLUP and MLP, we obtain

$$\pi_{r,s:n} = \Pr\left(|(X_{s:n} - X_{r:n}) - \sigma\pi_0| < |(X_{s:n} - X_{r:n}) - \sigma\pi_1|\right). \quad (3.6)$$

Squaring both sides of inequality and simplifying, (3.6) becomes

$$\Pr\left(2(X_{s:n} - X_{r:n})\sigma(\pi_1 - \pi_0) < \sigma^2({\pi^2}_1 - {\pi_0}^2)\right). \quad (3.7)$$

We have $\pi_1 > \pi_0$, so simplifying (3.7) gives

$\Pr\left(2(X_{s:n} - X_{r:n}) < \sigma(\pi_1 + \pi_0)\right)$.

Dividing by 2σ, we get

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Pr $\left(U < \frac{\pi_1 + \pi_0}{2}\right) = 1 - \Pr\left(U > \frac{\pi_1 + \pi_0}{2}\right)$, where U given in Lemma 3.3. (3.8)

Application of Lemma 3.3, binomial expression, and then integration gives:

$$\pi_{r,s:n} = 1 - \frac{(n-r)!}{(s-r-1)!(n-s)!} \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \frac{e^{-D(n-s+i+1)}}{n-s+i+1}.$$

For n=10, we display the values of $\pi_{r,s,n}$ given in Theorem 3.2. These values represented in Table 3 reveal that the BLUP is Pitman-closer than the MLP for all choices of r and s.

Table 3: Pitman Closeness Probability of MLP verses BLUP, σ is known, n=10.

r/s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.39	0.43	0.46	0.46	0.46	0.47	0.46	0.46	0.47
2		0.38	0.44	0.45	0.45	0.47	0.46	0.47	0.46
3			0.38	0.44	0.45	0.46	0.46	0.47	0.47
4				0.39	0.44	0.45	0.46	0.46	0.46
5					0.39	0.44	0.45	0.45	0.46
6						0.39	0.43	0.44	0.45
7							0.39	0.43	0.45
8								0.39	0.44
9									0.39

Theorem 3.3: Let $X_{1:n}$, $X_{2:n}$, ..., $X_{r:n}$ be a sample from Exp(σ), then the desired Pitman closeness probability of MLP and BLUP as a predictors of $X_{s:n}$, when σ is unknown, is

$$\pi_{r,s:n} = 1 - \frac{(n-r)!}{(s-r-1)!(n-s)!} \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \left(\frac{1}{n-s+i+1}\right)\left(\frac{1}{\{1+(n-s+i+1)D\}^r}\right),$$

where D=$\frac{\left(\frac{\pi_1}{r} + \frac{\pi_0}{r+1}\right)}{2}$.

Proof: From (2.7) and (2.8) we have, when σ is unknown,

$$\delta_{MLP} = X_{r:n} + \frac{T}{r+1}\pi_0, \delta_{BLUP} = X_{r:n} + \frac{T}{r}\pi_1.$$

Then, the desired Pitman closeness probability becomes

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$\pi_{r,s:n}$ =Pr ($|X_{s:n}-\delta_{MLP}|<|X_{s:n}-\delta_{BLUP}|$).

Upon using the expression of the MLP and BLUP given above, we obtain

$$\pi_{r,s:n}=\Pr\left(\left|(X_{s:n}-X_{r:n})-\frac{T}{r+1}\pi_0\right|<\left|(X_{s:n}-X_{r:n})-\frac{T}{r}\pi_1\right|\right). \quad (3.9)$$

Squaring both sides of the inequality and rearranging, (3.9) becomes

$$\Pr\left(2(X_{s:n}-X_{r:n})T\left(\frac{\pi_1}{r}-\frac{\pi_0}{r+1}\right)<T^2\left(\frac{\pi^2_{\ 1}}{r^2}-\frac{\pi^2_{\ 0}}{(r+1)^2}\right)\right). \quad (3.10)$$

Since $\pi_1>\pi_0$ and $\Pr(\mathrm{T}>0)=1$, (3.10) reduces to

$$\Pr\left(2(X_{s:n}-X_{r:n})<T\left(\frac{\pi_1}{r}+\frac{\pi_0}{r+1}\right)\right),$$

Dividing by 2σ, we have

$\pi_{r,s:n}=1-\Pr\left(U>W\ \mathrm{D}\right)$, where U and W are given in Lemma 3.3 and Lemma 1.5, respectively, and $D=\frac{1}{2}\left(\frac{\pi_1}{r}+\frac{\pi_0}{r+1}\right)$.

Using steps similar to prove Theorem 3.1, the result is established.

From Tables 4 and 5 we can read the values of $\pi_{r,s:n}$ between the BLUP and MLP, when σ is unknown, for n=10 and n=15. These values showed that the BLUP is Pitman-closer to $X_{s:n}$ than MLP for all r and s.

Table 4: Pitman Closeness Probability of MLP verses BLUP, σ is unknown, n=10.

r\s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.33	0.30	0.29	0.28	0.28	0.28	0.27	0.27	0.27
2		0.36	0.35	0.34	0.33	0.33	0.32	0.32	0.32
3			0.37	0.37	0.36	0.36	0.35	0.35	0.35
4				0.37	0.38	0.38	0.37	0.37	.37
5					0.37	0.39	0.39	0.39	0.38
6						0.38	0.40	0.40	0.40
7							0.38	0.40	0.40
8								0.38	0.41
9									0.38

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Table 5: Pitman Closeness Probability of MLP verses BLUP, σ is unknown, n=15.

r\s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
1	0.33	0.30	0.29	0.28	0.28	0.28	0.27	0.27	0.27	0.27	0.27	0.27	0.27	0.27
2		0.36	0.35	0.34	0.33	0.33	0.32	0.32	0.32	0.32	0.32	0.32	0.31	0.31
3			0.37	0.37	0.36	0.36	0.35	0.35	0.35	0.35	0.35	0.34	0.34	0.34
4				0.37	0.38	0.38	0.37	0.37	0.37	0.37	0.36	0.36	0.36	0.36
5					0.37	0.39	0.39	0.39	0.38	0.38	0.38	0.38	0.38	0.37
6						0.38	0.40	0.40	0.40	0.39	0.39	0.39	0.39	0.38
7							0.38	0.40	0.40	0.40	0.40	0.40	0.40	0.39
8								0.38	0.41	0.41	0.41	0.41	0.40	0.40
9									0.38	0.41	0.41	0.41	0.41	0.41
10										0.38	0.41	0.42	0.42	0.41
11											0.38	0.41	0.42	0.42
12												0.38	0.42	0.42
13													0.38	0.42
14														0.38

3.3 MLP vs. BLIP

Theorem 3.4: Let $X_{1:n}$, $X_{2:n}$, ..., $X_{r:n}$ be a sample from Exp(σ), then the desired Pitman closeness probability of MLP and BLIP as a predictor of $X_{s:n}$, when σ is unknown, is

$$\pi_{r,s:n} = 1 - \frac{(n-r)!}{(s-r-1)!(n-s)!} \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \left(\frac{1}{n-s+i+1} \right) \left(\frac{1}{\{1+(n-s+i+1)D\}^r} \right)$$

where, $D = \dfrac{(\pi_0 + \pi_1)}{2(r+1)}$.

Proof: From (2.7) and (2.10) , we have

$\delta_{MLP} = X_{r:n} + \dfrac{T}{r+1}\pi_0$ and $\delta_{BLIP} = X_{r:n} + \dfrac{T}{r+1}\pi_1$.

So, the desired Pitman closeness probability becomes

$\pi_{r,s:n}$ =Pr ($|X_{s:n} - \delta_{MLP}| < |X_{s:n} - \delta_{BLIP}|$).

$$= \Pr\left(\left| (X_{s:n} - X_{r:n}) - \frac{T}{r+1}\pi_0 \right| < \left| (X_{s:n} - X_{r:n}) - \frac{T}{r+1}\pi_1 \right| \right). \qquad (3.11)$$

Squaring both sides of inequality and simplifying, (3.11) becomes

$$\Pr\left(2(X_{s:n} - X_{r:n}) \frac{T}{r+1}(\pi_1 - \pi_0) < \frac{T^2}{(r+1)^2}({\pi^2}_1 - {\pi^2}_0) \right). \qquad (3.12)$$

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Since $\pi_1 > \pi_0$ and $\Pr(\, T > 0) = 1$, the inequality (3.12) reduces to

$$1\text{-Pr}\left((X_{s:n} - X_{r:n}) > \frac{T}{2}\left(\frac{\pi_1 + \pi_0}{r+1}\right) \right),$$

Dividing by 2σ, we have

$$\pi_{r,s:n} = 1\text{-Pr } (U > W \;\, D), \tag{3.13}$$

where $D = \dfrac{\pi_1 + \pi_0}{2(r+1)}$.

Applying Theorem 3.1, the result is obtained.

The values of $\pi_{r,s:n}$ between MLP and BLIP, when σ is unknown, for n=10 and n=15 are displayed in Table 6 and 7, respectively. We find that BLIP is Pitman –closer to $X_{s:n}$ than MLP for all r and s.

Table 6: Pitman Closeness Probability of MLP verses BLIP, σ is unknown, n=10.

r/s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.2	0.18	0.17	0.16	0.15	0.15	0.15	0.15	0.15
2		0.26	0.25	0.24	0.23	0.23	0.22	0.22	0.22
3			0.29	0.30	0.29	0.28	0.27	0.27	0.27
4				0.31	0.32	0.31	0.31	0.30	0.30
5					0.32	0.34	0.33	0.33	0.32
6						0.33	0.35	0.35	0.35
7							0.34	0.36	0.36
8								0.35	0.37
9									0.35

Table 7: Pitman Closeness Probability of MLP verses BLIP, σ is unknown, n=15.

r/s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
1	0.2	0.18	0.17	0.16	0.15	0.15	0.15	0.15	0.14	0.14	0.14	0.14	0.14	0.14
2		0.26	0.25	0.24	0.23	0.23	0.22	0.22	0.22	0.22	0.21	0.21	0.21	0.21
3			0.29	0.30	0.29	0.28	0.27	0.27	0.26	0.26	0.26	0.26	0.25	0.25
4				0.31	0.32	0.31	0.31	0.30	0.30	0.29	0.29	0.29	0.29	0.28
5					0.32	0.34	0.33	0.33	0.32	0.32	0.32	0.31	0.31	0.31
6						0.33	0.35	0.35	0.34	0.34	0.34	0.33	0.33	0.33
7							0.34	0.36	0.36	0.36	0.35	0.35	0.35	0.34
8								0.35	0.37	0.37	0.37	0.36	0.36	0.36
9									0.35	0.38	0.38	0.37	0.37	0.37
10										0.35	0.38	0.38	0.38	0.38
11											0.36	.39	0.39	0.39
12												0.36	0.39	0.39
13													0.36	0.40
14														0.36

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3.4 MLP vs. CMP

Theorem 3.5: Let $X_{1:n}$, $X_{2:n}$, ..., $X_{r:n}$ be a sample from Exp(σ), then the desired Pitman closeness probability of MLP and CMP as predictors of $X_{s:n}$, when σ is known, is

$$\pi_{r,s:n} = 1 - \frac{(n-r)!}{(s-r-1)!(n-s)!} \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \frac{e^{-D(n-s+i+1)}}{n-s+i+1},$$

where D=$\frac{\pi_0 + m}{2}$.

Proof: Immediately, when we apply the Pitman closeness probability on the MLP and CMP that are given in (2.7) and (2.11), respectively, we get

$$\pi_{r,s:n} = \Pr\left(\left|(X_{s:n} - X_{r:n}) - \sigma\pi_0\right| < \left|(X_{s:n} - X_{r:n}) - \sigma m\right|\right).$$

(3.14)

Squaring both sides of the inequality and simplifying, (3.14) becomes

$$\Pr\left(2(X_{s:n} - X_{r:n})\sigma(m - \pi_0) < \sigma^2(m^2 - \pi_0{}^2)\right). \tag{3.15}$$

Now since $\pi_0 < m$, inequality (3.15) reduces to

$$\Pr\left(2(X_{s:n} - X_{r:n}) < \sigma(\pi_0 + m)\right).$$

Dividing by 2σ, to obtain

$$1\text{-}\Pr\left(U > D\right). \tag{3.16}$$

Applying Lemma 3.3, we obtain

$$\pi_{r,s:n} = 1 - \frac{(n-r)!}{(s-r-1)!(n-s)!} \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \frac{e^{-D(n-s+i+1)}}{n-s+i+1},$$

where D=$\frac{\pi_0 + m}{2}$.

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The values of Pitman closeness probabilities, when σ is known, for n=10 are presented in Table 8. These results reveal that the CMP is Pitman-closer than the MLP for all choices of r and s.

Table8: Pitman Closeness Probability of MLP verses CMP, σ is known, n=10.

r\s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.29	0.38	0.41	0.42	0.43	0.43	0.43	0.43	0.42
2		0.29	0.38	0.41	0.42	0.43	0.43	0.43	0.42
3			0.29	0.38	0.41	0.42	0.43	0.43	0.42
4				0.29	0.38	0.41	0.42	0.42	0.41
5					0.29	0.38	0.41	0.42	0.41
6						0.29	0.38	0.40	0.41
7							0.29	0.38	0.40
8								0.29	0.38
9									0.29

Theorem 3.6: Let $X_{1:n}$, $X_{2:n}$, ..., $X_{r:n}$ be a sample from Exp(σ), then the desired Pitman closeness probability of MLP and CMP as predictors of $X_{s:n}$, when σ is unknown, is

$$\pi_{r,s:n} = 1 - \frac{(n-r)!}{(s-r-1)!(n-s)!}\sum_{i=0}^{s-r-1}(-1)^{i}\binom{s-r-1}{i}\left(\frac{1}{n-s+i+1}\right)\left(\frac{1}{\{1+(n-s+i+1)D\}^{r}}\right),$$

where $D=\frac{1}{2}\left(\frac{m}{r}+\frac{\pi_0}{r+1}\right)$.

Proof: Upon using the expression of the MLP and CMP that are given in (2.7) and (2.12), respectively, we have

$$\pi_{r,s:n} = \Pr\left(\left|(X_{s:n}-X_{r:n})-\frac{T}{r+1}\pi_0\right| < \left|(X_{s:n}-X_{r:n})-\frac{T}{r}m\right|\right). \qquad (3.17)$$

Squaring both sides of the inequality and rearranging, (3.17) becomes

$$\Pr\left(2(X_{s:n}-X_{r:n})T\left(\frac{m}{r}-\frac{\pi_0}{r+1}\right) < T^{2}\left(\frac{m^{2}}{r^{2}}-\frac{\pi^{2}_{0}}{(r+1)^{2}}\right)\right). \qquad (3.18)$$

Since $m > \pi_0$ and $\Pr(T > 0) = 1$, (3.18) reduces to

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$$\Pr\left(2(X_{s:n}-X_{r:n})<T\,(\frac{m}{r}+\frac{\pi_0}{r+1})\right).$$

Dividing by 2σ, we have

$$\Pr\ 1-\Pr\left(U>W\ \ D\right), \tag{3.19}$$

Applying Theorem 3.1, the result is established.

Tables 9 and 10 present the values of Pitman closeness probabilities between MLP and CMP, when σ is unknown, for n=10 and n=15, respectively. These results reveal that the CMP is Pitman –closer to $X_{s:n}$ than MLP for all r and s.

Table 9: Pitman Closeness Probability of MLP verses CMP, σ is unknown, n=10.

r/s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.25	0.27	0.27	0.26	0.26	0.26	0.26	0.26	0.25
2		0.27	0.30	0.31	0.31	0.31	0.31	0.30	0.30
3			0.27	0.32	0.33	0.33	0.33	0.33	0.32
4				0.28	0.33	0.34	0.35	0.34	0.34
5					0.28	0.34	0.35	0.35	0.35
6						0.28	0.35	0.36	0.35
7							0.28	0.35	0.36
8								0.28	0.35
9									0.28

Table10: Pitman Closeness Probability of MLP verses CMP, σ is unknown, n=15.

r/s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
1	0.25	0.27	0.27	0.27	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.25
2		0.27	0.30	0.31	0.31	0.31	0.31	0.31	0.31	0.31	0.30	0.30	0.30	0.29
3			0.27	0.32	0.33	0.33	0.33	0.33	0.33	0.33	0.33	0.33	0.32	0.32
4				0.28	0.33	0.34	0.35	0.35	0.35	0.35	0.35	0.34	0.34	0.33
5					0.28	0.34	0.35	0.36	0.36	0.36	0.36	0.36	0.35	0.34
6						0.28	0.35	0.36	0.37	0.37	0.37	0.37	0.36	0.35
7							0.28	0.35	0.37	0.37	0.37	0.37	0.37	0.36
8								0.28	0.35	0.37	0.38	0.38	0.37	0.37
9									0.28	0.36	0.37	0.38	0.38	0.37
10										0.28	0.36	0.38	0.38	0.37
11											0.28	0.36	0.38	0.37
12												0.28	0.36	0.37
13													0.28	0.36
14														0.28

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3.5 BLUP vs. CMP

Theorem 3.7: Let $X_{1:n}$, $X_{2:n}$, ..., $X_{r:n}$ be a sample from Exp(σ), then the desired Pitman closeness probability of BLUP and CMP as predictors of $X_{s:n}$, when σ is known, is

$$\pi_{r,s:n} = \frac{(n-r)!}{(s-r-1)!(n-s)!}\sum_{i=0}^{s-r-1}(-1)^i\binom{s-r-1}{i}\frac{e^{-D(n-s+i+1)}}{n-s+i+1},$$

where D=$\dfrac{\pi_1+m}{2}$.

Proof: Analogous to the proof of Theorem 3.2.

For n=10, we computed the values of Pitman closeness probabilities $\pi_{r,s:n}$ between BLUP and CMP, when σ is known. These values are presented in Table 11.The results reveal that the CMP is Pitman-closer to $X_{s:n}$ than the BLUP for all choices of r and s.

Table 11: Pitman Closeness Probability of BLUP verses CMP, σ is known, n=10.

r/s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.42	0.45	0.46	0.46	0.46	0.46	0.46	0.46	0.45
2		0.42	0.44	0.46	0.46	0.46	0.46	0.46	0.45
3			0.42	0.45	0.45	0.46	0.46	0.46	0.45
4				0.42	0.45	0.45	0.46	0.46	0.45
5					0.42	0.45	0.45	0.46	0.45
6						0.42	0.45	0.45	0.45
7							0.42	0.44	0.45
8								0.42	0.44
9									0.42

Theorem 3.8: Let $X_{1:n}$, $X_{2:n}$, ..., $X_{r:n}$ be a sample from Exp(σ), then the desired Pitman closeness probability of BLUP and CMP as predictors of $X_{s:n}$, when σ is unknown, is

$$\pi_{r,s:n} = \frac{(n-r)!}{(s-r-1)!(n-s)!}\sum_{i=0}^{s-r-1}(-1)^i\binom{s-r-1}{i}\left(\frac{1}{n-s+i+1}\right)\left(\frac{1}{\{1+(n-s+i+1)D\}^r}\right),$$

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where D=$\frac{(m+\pi_1)}{2r}$.

Proof: Analogous to the proof of Theorem 3.3

The values of Pitman closeness probabilities, when σ is unknown, for n=10 and 15 are presented in Tables 12 and 13, respectively. They reveal that the BLUP is Pitman-closer to $X_{s:n}$ than the CMP when r=1. The CMP is always Pitman-closer when s=r+1 except for r=1.In general, for small r, the BLUP is Pitman-closer while the CMP is Pitman-closer for large values of r.

Table 12: Pitman Closeness Probability of BLUP verses CMP, σ is unknown, n=10.

r\s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.54	0.58	0.59	0.60	0.60	0.61	0.61	0.61	0.61
2		0.49	0.53	0.54	0.55	0.56	0.56	0.56	0.56
3			0.47	0.50	0.52	0.53	0.54	0.54	0.54
4				0.46	0.49	0.51	0.52	0.52	0.52
5					0.45	0.48	0.50	0.51	0.51
6						0.45	0.48	0.49	0.49
7							0.44	0.47	0.48
8								0.44	0.47
9									0.44

Table13: Pitman Closeness Probability of BLUP verses CMP, σ is unknown, n=15.

r\s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
1	0.54	0.58	0.59	0.60	0.60	0.61	0.61	0.61	0.61	0.61	0.62	0.62	0.62	0.61
2		0.49	0.53	0.54	0.55	0.56	0.56	0.57	0.57	0.57	0.57	0.57	0.57	0.57
3			0.47	0.50	0.52	0.53	0.54	0.54	0.54	0.55	0.55	0.55	0.55	0.55
4				0.46	0.49	0.51	0.52	0.52	0.53	0.53	0.53	0.54	0.54	0.53
5					0.45	0.48	0.50	0.51	0.51	0.52	0.52	0.52	0.52	0.52
6						0.45	0.48	0.49	0.50	0.51	0.51	0.51	0.52	0.51
7							0.44	0.47	0.49	0.50	0.50	0.51	0.51	0.50
8								0.44	0.47	0.48	0.49	0.50	0.50	0.50
9									0.44	0.47	0.48	0.49	0.49	0.49
10										0.44	0.47	0.48	0.48	0.48
11											0.44	0.46	0.48	0.48
12												0.44	0.46	0.47
13													0.44	0.46
14														0.43

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3.6 CMP vs. BLIP

Theorem 3.9: Let $X_{1:n}$, $X_{2:n}$, ..., $X_{r:n}$ be a sample from Exp(σ), then the desired Pitman closeness probability of CMP and BLIP of $X_{s:n}$, when σ is unknown, is

$$\pi_{r,s:n} = \begin{cases} 1-\dfrac{(n-r)!}{(s-r-1)!(n-s)!}\displaystyle\sum_{i=0}^{s-r-1}(-1)^i\binom{s-r-1}{i}\left(\dfrac{1}{n-s+i+1}\right)\left(\dfrac{1}{\{1+(n-s+i+1)D\}^r}\right), \\ (\dfrac{\pi_1}{r+1}-\dfrac{m}{r})>0. \\ \dfrac{(n-r)!}{(s-r-1)!(n-s)!}\displaystyle\sum_{i=0}^{s-r-1}(-1)^i\binom{s-r-1}{i}\left(\dfrac{1}{n-s+i+1}\right)\left(\dfrac{1}{\{1+(n-s+i+1)D\}^r}\right), \\ (\dfrac{\pi_1}{r+1}-\dfrac{m}{r})<0. \end{cases}$$

where D$=\frac{1}{2}(\frac{\pi_1}{r+1}+\frac{m}{r})$.

Proof: We have

$$\pi_{r,s:n} = \Pr\left(\left|(X_{s:n}-X_{r:n})-\frac{T}{r}m\right| < \left|(X_{s:n}-X_{r:n})-\frac{T}{r+1}\pi_1\right|\right). \tag{3.20}$$

Squaring the inequality and rearranging its sides, we get

$$\pi_{r,s:n} = \Pr\left(2(X_{s:n}-X_{r:n})(\frac{\pi_1}{r+1}-\frac{m}{r}) < T(\frac{\pi_1}{r+1}-\frac{m}{r})(\frac{\pi_1}{r+1}+\frac{m}{r})\right).$$

Since $\pi_1 > m$, and $(\frac{\pi_1}{r+1}-\frac{m}{r})$ not always positive, we have after diving by 2σ that

$$\pi_{r,s:n} = \begin{cases} 1-\Pr\left(U > W\,\dfrac{(\frac{\pi_1}{r+1}+\frac{m}{r})}{2}\right), (\dfrac{\pi_1}{r+1}-\dfrac{m}{r})>0. \\ \Pr\left(U > W\,\dfrac{(\frac{\pi_1}{r+1}+\frac{m}{r})}{2}\right), (\dfrac{\pi_1}{r+1}-\dfrac{m}{r})<0. \end{cases}$$

Applying the result of Theorem 3.1, we can express $\pi_{r,s:n}$ as

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$$\pi_{r,s:n}=\begin{cases}1-\dfrac{(n-r)!}{(s-r-1)!(n-s)!}\displaystyle\sum_{i=0}^{s-r-1}(-1)^{i}\binom{s-r-1}{i}\left(\frac{1}{n-s+i+1}\right)\left(\frac{1}{\{1+(n-s+i+1)D\}^{r}}\right),\\ (\dfrac{\pi_1}{r+1}-\dfrac{m}{r})>0.\\ \dfrac{(n-r)!}{(s-r-1)!(n-s)!}\displaystyle\sum_{i=0}^{s-r-1}(-1)^{i}\binom{s-r-1}{i}\left(\frac{1}{n-s+i+1}\right)\left(\frac{1}{\{1+(n-s+i+1)D\}^{r}}\right),\\ (\dfrac{\pi_1}{r+1}-\dfrac{m}{r})<0.\end{cases}$$

Where $D=\frac{1}{2}(\frac{\pi_1}{r+1}+\frac{m}{r})$.

The values of Pitman closeness probabilities of CMP and BLIP, when σ is known, are presented in Table 14 and 15, respectively. We are notice from this values that the CMP is Pitman –closer to $X_{s:n}$ than BLIP for most values of r and s except (r,s)=(3,4) and (r,s)=(4,5).

Table14: Pitman Closeness Probability of CMP verses BLIP, σ is unknown, n=10.

r\s	2	3	4	5	6	7	8	9	10
1	0.62	0.67	0.69	0.70	0.70	0.71	0.71	0.71	0.71
2		0.55	0.23	0.62	0.63	0.64	0.65	0.65	0.65
3			0.47	0.56	0.59	0.60	0.61	0.61	0.61
4				0.49	0.54	0.56	0.58	0.58	0.58
5					0.50	0.53	0.55	0.56	0.56
6						0.51	0.47	0.53	0.54
7							0.52	0.48	0.47
8								0.52	0.49
9									0.53

Table15: Pitman Closeness Probability of CMP verses BLIP, σ is unknown, n=15.

r\s	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0.62	0.67	0.69	0.70	0.70	0.71	0.71	0.71	0.71	0.72	0.72	0.72	0.72	0.72
2		0.55	0.60	0.62	0.63	0.64	0.65	0.65	0.66	0.66	0.66	0.66	0.66	0.66
3			0.47	0.56	0.59	0.60	0.61	0.62	0.62	0.62	0.63	0.63	0.63	0.62
4				0.49	0.54	0.56	0.58	0.59	0.59	0.60	0.60	0.60	0.60	0.60
5					0.50	0.53	0.55	0.56	0.57	0.58	0.58	0.58	0.58	0.58
6						0.51	0.47	0.54	0.55	0.56	0.56	0.57	0.57	0.56
7							0.52	0.10	0.53	0.54	0.55	0.55	0.55	0.55
8								0.52	0.49	0.47	0.53	0.54	0.54	0.54
9									0.53	0.49	0.48	0.52	0.53	0.53
10										0.53	0.50	0.48	0.47	0.47
11											0.54	0.50	0.49	0.49
12												0.54	0.51	0.50
13													0.54	0.51
14														0.54

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3.7 Summary and Conclusions

We are observed from all tables presented in the chapter that predictors have different PMC probabilities. Hence, we can order these predictors and determine which one is the best under PMC and conclude that

- CMP is the Pitman closer to $X_{s:n}$ than BLIP, MLP.
- BLUP is Pitman closer than BLIP and CMP for small r.
- BLUP is Pitman closer than MLP.
- MLP is the worst predictor under PMC.

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CHATER FOUR

Pitman Closeness Comparisons between Modified Predictors

4.1 Modified Maximum Likelihood Predictor.

4.2 Modified Maximum Likelihood Predictor verses Best Linear Unbiased Predictor.

4.3 Modified Maximum Likelihood Predictor verses Best Linear Invariant Predictor.

4.4 Modified Maximum Likelihood Predictor verses Conditional Median Predictor.

4.5 Summary and Conclusions.

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In this chapter, we will modify the maximum likelihood predictor to be an unbiased predictor. We will study the effect of the unbiasedness property on the values of Pitman Measure closeness probabilities that we computed in the previous chapter. We will table these values to notice the differences may occur.

4.1 Modified Maximum Likelihood Predictor (MMLP)

To modify the MLP to be an unbiased predictor, we will computed the bias and subtract it from this predictor. We will denote the MLP after modification as modified maximum likelihood predictor (MMLP). The following definitions are needed to find MMLP.

Definition 4.1: $\hat{X}_{s:n}$ is said to be an unbiased predictor of $X_{s:n}$ iff $E(\hat{X}_{s:n}) = E(X_{s:n})$.

Definition 4.2: If $\hat{X}_{s:n}$ is a predictor of $X_{s:n}$, then the bias in this predictor is equal to $\mathrm{b} = E(\hat{X}_{s:n}) - E(X_{s:n})$.

Modify the MLP for the σ known and unknown cases:

i) σ known

We know that the MLP in this case equal to $\delta_{MLP} = \hat{X}_{s:n} = X_{r:n} + \sigma\pi_0$, then the bias of this predictor is equal to

$$\mathrm{b}=\left(E(\hat{X}_{s:n}) - E(X_{s:n})\right) = E(X_{r:n}) + \sigma\pi_0 - E(X_{s:n}) = \sigma(\pi_0 - \pi_1) \tag{4.1}$$

Let

$$\begin{aligned}\delta_{MMLP} = \hat{X}_{s:n} &= \delta_{MLP} - b = \delta_{MLP} - \sigma(\pi_0 - \pi_1)\\ &= X_{r:n} + \sigma\pi_0 - \sigma(\pi_0 - \pi_1)\\ &= X_{r:n} + \sigma\pi_1 . \end{aligned} \tag{4.2}$$

We notice that the MMLP is the same as the BLUP when σ is known. So, no need for any new comparison between MMLP and other predictors.

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ii) σ unknown

In this case, the MLP in (2.7) is given by $\delta_{MLP} = \hat{X}_{s:n} = X_{r:n} + \frac{T}{r+1}\pi_0$. And the bias $b = \frac{r\sigma\pi_0}{r+1} - \sigma\pi_1$.

Let $\delta_{MMLP} = \delta_{MLP} - b = X_{r:n} + \frac{\pi_0}{r+1}T - \frac{r\sigma\pi_0}{r+1} + \sigma\pi_1$

$$= \frac{\pi_0}{r+1}(T - r\sigma) + \sigma\pi_1. \tag{4.3}$$

Hence, this predictor is now ready to be compared with other predictor under PMC.

4.2 MMLP vs. BLUP

Theorem 4.1: Let δ_{MMLP} be the maximum likelihood predictor corrected for bias and δ_{BLUP} be the best linear unbiased predictor for $X_{s:n}$, then the Pitman closeness probability for δ_{MMLP} and δ_{BLUP} is given by

$$\pi_{r,s:n} = b_{r,s,n} \sum_{i=0}^{s-r-1} \frac{(-1)\binom{s-r-1}{i}}{(n-s+i+1)} [1 - G(r,1,r) + (\tfrac{1}{2}(n-s+i+1)k_1 + 1)^{-r} e^{\frac{-1}{2}k_2 r(n-s+i+1)} \{2G(r,1,\tfrac{1}{2}r(n-s+i+1)k_1 + r) - 1\}],$$

where $G(\alpha,\beta;x)$ is the gamma cumulative distribution function with shape parameter α and scale parameter β, evaluated at x. $k_1 = (\frac{\pi_0}{r+1} + \frac{\pi_1}{r})$, and $k_2 = (\frac{\pi_1}{r} - \frac{\pi_0}{r+1})$.

Proof: We have, by (4.3) and (2.8), that

$\delta_{MMLP} = X_{r:n} + \frac{\pi_0}{r+1}(T - r\sigma) + \sigma\pi_1$, and $\delta_{BLUP} = X_{r:n} + \frac{\pi_1}{r}T$.

Then, the desired Pitman closeness probability becomes

$$\pi_{r,s:n} = \Pr\left(\left|(X_{s:n} - X_{r:n}) - \frac{\pi_0}{r+1}(T - r\sigma) - \sigma\pi_1\right| < \left|(X_{s:n} - X_{r:n}) - \frac{T}{r}\pi_1\right|\right). \tag{4.4}$$

Dividing both sides of (4.4) by σ, one obtains

$$\pi_{r,s:n} = \Pr(|U - g_1(W)| < |U - g_2(W)|), \tag{4.5}$$

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where $g_1(W) = \frac{\pi_0}{r+1}(W - r) + \pi_1$ and $g_2(W) = \frac{\pi_1}{r} W$. U and W as defined in Theorem 3.3.

Squaring both sides of the inequality and simplifying, (4.5) becomes

$$\pi_{r,s:n} = \Pr\left(2U\left(g_2(W) - g_1(W)\right) < g_2^{\,2}(W) - g_1^{\,2}(W)\right)$$

$$= \Pr\left(2U < g_1(W) + g_2(W), g_2(W) - g_1(W) > 0\right) + \Pr\left(2U > g_1(W) + g_2(W), g_2(W) - g_1(W) < 0\right).$$

But we have $g_2(W) - g_1(W) > 0$ iff $(\frac{\pi_1}{r} - \frac{\pi_0}{r+1})\, W - r\, (\frac{\pi_1}{r} - \frac{\pi_0}{r+1}) > 0.$

Since $(\frac{\pi_1}{r} - \frac{\pi_0}{r+1}) > 0$, then $g_2(W) - g_1(W) > 0$ iff $W > r$.

Also, $g_2(W) + g_1(W) = k_1 W + k_2 r$, where $k_1 = (\frac{\pi_0}{r+1} + \frac{\pi_1}{r})$ and $k_2 = (\frac{\pi_1}{r} - \frac{\pi_0}{r+1})$.

So, $\pi_{r,s:n}$ simplifies to,

$$\Pr(2U < k_1 W + k_2 r,\ W > r) + \Pr(2U > k_1 W + k_2 r,\ W < r)$$

$$= \int_0^\infty [\Pr(W > Max(r, \frac{2u - k_2 r}{k_1}) + \Pr(W < Min(r, \frac{2u - k_2 r}{k_1}))] f_U(u)\, du.$$

$$= \int_0^{\frac{1}{2}r(k_1+k_2)} \int_r^\infty f_W(w) f_U(u)\, dw\ du + \int_{\frac{1}{2}r(k_1+k_2)}^\infty \int_{\frac{2u-k_2 r}{k_1}}^\infty f_W(w) f_U(u)\, dw\ du +$$

$$\int_{\frac{1}{2}r(k_1+k_2)}^\infty \int_0^r f_W(w) f_U(u)\, dw\ du + \int_0^{\frac{1}{2}r(k_1+k_2)} \int_0^{\frac{2u-k_2 r}{k_1}} f_W(w) f_U(u)\, dw\ du.$$

Combining and changing the order of integration, we obtain

$$\pi_{r,s:n} = \int_0^r \int_{\frac{1}{2}(k_1 w + k_2 r)}^\infty f_W(w) f_U(u)\, dw\ du + \int_r^\infty \int_0^{\frac{1}{2}(k_1 w + k_2 r)} f_W(w) f_U(u)\, dw\ du.$$

Now, the density of U and W are given in Lemma (3.3) and in Lemma (1.5), respectively. Substituting these densities in the above terms and following steps similar to those in proving Theorem 3.1, the result is established.

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Tables 16 and 17 display the values of Pitman closeness probabilities when n=10 and 15, respectively. These results showed that the MMLP is Pitman-closer to $X_{s:n}$ than the BLUP for all choices of r and s

Table 16: Pitman Closeness Probability of MMLP verses BLUP, σ is unknown, n=10.

r/s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.59	0.69	0.74	0.78	0.80	0.82	0.83	0.83	0.82
2		0.57	0.66	0.71	0.74	0.76	0.78	0.78	0.77
3			0.56	0.64	0.68	0.71	0.73	0.74	0.73
4				0.55	0.62	0.67	0.69	0.71	0.70
5					0.54	0.61	0.65	0.68	0.68
6						0.54	0.60	0.64	0.65
7							0.54	0.60	0.62
8								0.53	0.59
9									0.53

Table 17: Pitman Closeness Probability of MMLP verses BLUP, σ is unknown, n=15.

r/s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
1	0.59	0.69	0.75	0.78	0.80	0.82	0.84	0.85	0.85	0.86	0.86	0.86	0.86	0.84
2		0.57	0.66	0.71	0.74	0.77	0.78	0.80	0.81	0.82	0.82	0.82	0.82	0.80
3			0.56	0.64	0.69	0.72	0.74	0.76	0.77	0.78	0.79	0.79	0.79	0.77
4				0.55	0.62	0.67	0.70	0.72	0.74	0.75	0.76	0.77	0.76	0.74
5					0.54	0.61	0.66	0.68	0.71	0.72	0.73	0.74	0.74	0.72
6						0.54	0.60	0.64	0.67	0.69	0.71	0.72	0.72	0.70
7							0.54	0.60	0.64	0.66	0.68	0.69	0.70	0.69
8								0.53	0.59	0.63	0.65	0.67	0.68	0.67
9									0.53	0.59	0.62	0.64	0.66	0.65
10										0.53	0.58	0.61	0.63	0.64
11											0.53	0.58	0.61	0.62
12												0.53	0.57	0.60
13													0.53	0.57
14														0.52

4.3 MMLP vs. BLIP

Theorem 4.2: Let δ_{MMLP} be the maximum likelihood predictor corrected for bias and δ_{BLIP} be the best linear invariant predictor for $X_{s:n}$. then, the Pitman closeness probability for δ_{MMLP} and δ_{BLIP} is given by

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$$\pi_{r,s:n} = b_{r,s,n} \sum_{i=0}^{s-r-1} \frac{(-1)\binom{s-r-1}{i}}{(n-s+i+1)} [1-G(r,1,c)+(\tfrac{1}{2}(n-s+i+1)k_3+1)^{-r} e^{\frac{-1}{2}k_4 r(n-s+i+1)}$$

$$\{2G(r,1,\tfrac{1}{2}c(n-s+i+1)k_3+c)-1\}],$$

where $k_3 = (\frac{\pi_0}{r+1}+\frac{\pi_1}{r+1})$, $k_4 = (\frac{\pi_1}{r}-\frac{\pi_0}{r+1})$, and $c = \dfrac{r(r+1)k_4}{\pi_1-\pi_0}$.

Proof: We have, by (4.3) and (2.10), that

$$\delta_{MMLP} = X_{r:n} + \frac{\pi_0}{r+1}(T-r\sigma)+\sigma\pi_1 \text{ and } \delta_{BLIP} = X_{r:n} + \frac{\pi_1}{r+1}T .$$

Thus,

$$\pi_{r,s:n} = \Pr(\left|(X_{s:n}-X_{r:n})-\frac{\pi_0}{r+1}(T-r\sigma)-\sigma\pi_1\right| < \left|(X_{s:n}-X_{r:n})-\frac{T}{r+1}\pi_1\right|). \tag{4.6}$$

Dividing both sides of (4.6) by σ, one obtains

$$\Pr(|U-g_1(\mathrm{W})| < |U-g_2(\mathrm{W})|), \tag{4.7}$$

where W and U are given in Theorem 4.1, and

$g_1(W) = \frac{\pi_0}{r+1}(W-r)+\pi_1$, $g_2(W) = \frac{\pi_1}{r+1}W$.

Squaring both sides of the inequality and simplifying, (4.7) becomes

$$\begin{aligned}\pi_{r,s:n} &= \Pr\left(2U\left(g_2(W)-g_1(W)\right) < g_2^{\,2}(W)-g_1^{\,2}(W)\right)\\ &= \Pr\left(2U < g_1(W)+g_2(W), g_2(W)-g_1(W)>0\right)+\\ &\qquad \Pr\left(2U > g_1(W)+g_2(W), g_2(W)-g_1(W)<0\right).\end{aligned}$$

We have $g_2(W)-g_1(W)>0$ iff $(\frac{\pi_1}{r+1}-\frac{\pi_0}{r+1})\, W - r\, (\frac{\pi_1}{r}-\frac{\pi_0}{r+1})>0.$

Since $(\frac{\pi_1}{r}-\frac{\pi_0}{r+1})>0$, then $g_2(W)-g_1(W)>0$ iff $W>c$, where $c = r\dfrac{(\frac{\pi_1}{r}-\frac{\pi_0}{r+1})}{(\frac{\pi_1-\pi_0}{r+1})}$.

Also, $g_2(W)+g_1(W) = k_3\, W + k_4 r$, where $k_3 = (\frac{\pi_0}{r+1}+\frac{\pi_1}{r+1})$

and $k_4 = (\frac{\pi_1}{r}-\frac{\pi_0}{r+1})$.

So, $\pi_{r,s:n}$ simplifies to,

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$\Pr(2U < k_3 W + k_4 r,\ W > c) + \Pr(2U > k_3 W + k_4 r,\ W < c)$. The rest of the proof is analogous to Theorem 4.1.

Table 18 and 19 display the values of Pitman closeness probabilities when n=10 and 15, respectively. These results showed that the MMLP is Pitman-closer than the BLIP for all choices of r and s.

Table 18: Pitman Closeness Probability of MMLP verses BLIP, σ is unknown, n=10

r/s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.54	0.62	0.65	0.67	0.69	0.70	0.71	0.71	0.70
2		0.52	0.59	0.61	0.62	0.62	0.63	0.63	0.63
3			0.51	0.57	0.59	0.59	0.59	0.60	0.60
4				0.51	0.56	0.58	0.58	0.58	0.58
5					0.50	0.55	0.57	0.57	0.57
6						0.50	0.55	0.56	0.56
7							0.55	0.54	0.55
8								0.50	0.54
9									0.50

Table 19: Pitman Closeness Probability of MMLP verses BLIP, σ is unknown, n=15.

r/s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
1	0.54	0.62	0.65	0.67	0.69	0.70	0.71	0.72	0.72	0.73	0.73	0.73	0.73	0.72
2		0.52	0.59	0.61	0.62	0.63	0.63	0.64	0.65	0.65	0.65	0.66	0.65	0.64
3			0.51	0.57	0.59	0.59	0.59	0.60	0.60	0.61	0.61	0.61	0.61	0.61
4				0.51	0.56	0.58	0.58	0.58	0.58	0.58	0.59	0.59	0.59	0.59
5					0.50	0.55	0.57	0.57	0.57	0.57	0.57	0.57	0.57	0.57
6						0.50	0.55	0.56	0.56	0.56	0.56	0.56	0.56	0.57
7							0.50	0.55	0.56	0.56	0.56	0.56	0.56	0.56
8								0.50	0.54	0.56	0.56	0.56	0.56	0.56
9									0.50	0.54	0.55	0.56	0.56	0.56
10										0.50	0.54	0.55	0.55	0.55
11											0.50	0.54	0.55	0.55
12												0.50	0.53	0.54
13													0.50	0.53
14														0.50

4.4 MMLP vs. CMP

Theorem 4.3: Let δ_{MMLP} be the maximum likelihood predictor corrected for bias and δ_{CMP} be the best median unbiased predictor for $X_{s:n}$ then, the Pitman closeness probability for δ_{MMLP} and δ_{CMP} is given by

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$$\pi_{r,s:n} = c(r,s,n)\sum_{i=0}^{s-r-1}\frac{(-1)\binom{s-r-1}{i}}{(n-s+i+1)}[1-G(r,1,c)+(\tfrac{1}{2}(n-s+i+1)k_5+1)^{-r}e^{\frac{-1}{2}k_6 r(n-s+i+1)}$$

$$\{2G(r,1,\tfrac{1}{2}c(n-s+i+1)k_5+c)-1\}],$$

where $G(\alpha,\beta,x)$ is the gamma cumulative distribution function with shape parameter α and scale parameter β , evaluated at x. $k_5 = (\frac{\pi_0}{r+1}+\frac{m}{r})$, $k_6 = (\frac{\pi_1}{r}-\frac{\pi_0}{r+1})$ and $c = \frac{k_6}{(\frac{m}{r}-\frac{\pi_0}{r+1})}$.

Proof: We have, by (4.3) and (2.12), that

$\delta_{MMLP} = X_{r:n} + \frac{\pi_0}{r+1}(T - r\sigma) + \sigma\pi_1$ and $\delta_{CMP} = X_{r:n} + \frac{\pi_1}{r}m$, respectively.

Thus,

$$\pi_{r,s:n} = \Pr\left(\left|(X_{s:n} - X_{r:n}) - \frac{\pi_0}{r+1}(T - r\sigma) - \sigma\pi_1\right| < \left|(X_{s:n} - X_{r:n}) - \frac{T}{r}m\right|\right) \tag{4.8}$$

Dividing both sides of (4.8) by σ , one obtains

$$\Pr(|U - g_1(W)| < |U - g_2(W)|), \tag{4.9}$$

where W and U are given in Theorem 3.3, and

$g_1(W) = \frac{\pi_0}{r+1}(W - r) + \pi_1$, $g_2(W) = \frac{W}{r}m$.

Squaring both sides of the inequality and rearranging, (4.9) becomes

$$\pi_{r,s:n} = \Pr\left(2U\left(g_2(W) - g_1(W)\right) < g_2^{\,2}(W) - g_1^{\,2}(W)\right)$$

$$= \Pr\left(2U < g_1(\mathrm{W}) + g_2(\mathrm{W}), g_2(\mathrm{W}) - g_1(\mathrm{W}) > 0\right) + \Pr\left(2U > g_1(\mathrm{W}) + g_2(\mathrm{W}), g_2(\mathrm{W}) - g_1(\mathrm{W}) < 0\right).$$

We have $g_2(W) - g_1(W) > 0$ iff $(\frac{m}{r} - \frac{\pi_0}{r+1})\,W - r\,(\frac{\pi_1}{r} - \frac{\pi_0}{r+1}) > 0$.

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Since $(\frac{\pi_1}{r}-\frac{\pi_0}{r+1})>0$ and $(\frac{m}{r}-\frac{\pi_0}{r+1})>0$, then $g_2(W)-g_1(W)>0$ iff $W>c$,

where $c=r\frac{(\frac{\pi_1}{r}-\frac{\pi_0}{r+1})}{(\frac{m}{r}-\frac{\pi_0}{r+1})}$.

Also, $g_2(W)+g_1(W)=k_5\,W+k_6 r$, where $k_5=(\frac{\pi_0}{r+1}+\frac{m}{r})$ and $k_6=(\frac{\pi_1}{r}-\frac{\pi_0}{r+1})$.

So, $\pi_{r,s:n}$ simplifies to,

$\Pr(2U<k_5\,W+k_6 r,\ W>c)+\Pr(2U>k_5\,W+k_6 r,\ W<c)$

Following steps similar to those in the proof of Theorem 4.1, the result is established.

We note from values of Pitman closeness probabilities which are presented in Table 20 and 21, respectively , that the MMLP is Pitman-closer to $X_{s:n}$ than the CMP for all (r, s) except for (r, r+1),r=4,…,n when n=10. The same conclusion is true for n=15, that is, MMLP is better than CMP for all (r, s) excluding the set

$\{(r, r+1), r=4,\ldots,15\}\cup\{(12,14),(13,15)\}$.

Table 20: Pitman Closeness Probability of MMLP verses CMP, σ is unknown , n=10

r/s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.55	0.67	0.73	0.77	0.80	0.81	0.82	0.83	0.81
2		0.52	0.63	0.69	0.72	0.75	0.76	0.77	0.75
3			0.50	0.60	0.65	0.69	0.71	0.72	0.70
4				0.49	0.57	0.62	0.66	0.67	0.66
5					0.48	0.56	0.60	0.62	0.62
6						0.47	0.54	0.58	0.58
7							0.47	0.53	0.54
8								0.46	0.51
9									0.46

Table 21: Pitman Closeness Probability of MMLP verses CMP, σ is unknown, n=15

r/s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
1	0.55	0.67	0.74	0.77	0.80	0.82	0.83	0.84	0.85	0.86	0.86	0.86	0.86	0.84
2		0.52	0.63	0.69	0.73	0.75	0.77	0.79	0.80	0.81	0.81	0.82	0.81	0.78
3			0.50	0.60	0.65	0.69	0.72	0.74	0.76	0.77	0.77	0.78	0.77	0.74
4				0.49	0.57	0.63	0.66	0.69	0.71	0.73	0.74	0.74	0.73	0.70
5					0.48	0.56	0.60	0.64	0.67	0.68	0.70	0.70	0.70	0.67
6						0.47	0.54	0.59	0.62	0.64	0.66	0.67	0.67	0.64
7							0.47	0.53	0.57	0.60	0.62	0.63	0.64	0.61
8								0.46	0.52	0.56	0.58	0.60	0.61	0.59
9									0.46	0.51	0.54	0.57	0.58	0.56
10										0.45	0.50	0.53	0.55	0.54

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11											0.45	0.50	0.52	0.52
12												0.45	0.49	0.50
13													0.45	0.48
14														0.44

4.5 Summary and Conclusions

After correcting the MLP for bias, we have computed the values of $\pi_{r,s,n}$. Depending on these values, we have observed that

- When σ is known MMLP is the same as BLUP.
- MMLP is Pitman closer to $X_{s:n}$ than BLUP and BLIP.
- MMLP is Pitman closer than CMP for all r, but few (r, s) pairs.
- Correcting the MLP for bias improves the PMC probabilities against other predictors under consideration significantly.

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مقارنات تقارب بيتمان لتنبؤات الإحصائيات الرتبية المستقبلية من التوزيع الأسي

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ملخّــــــــص

في هذه الرسالة تم استعراض التنبؤات الاحتمالية المستخدمة في التنبؤ عن الإحصائيات الرتبيّه المستقبلية للعينات المبتورة من الطرفين من النوع الثاني. حيث تمت مقارنة التنبــؤات بمقياس تقارب بيتمان وتم اشتقاق احتمالية تقارب بيتمان وحسابها رقميا لاكثر من اختيار. وقد تبيّن بأن التنبؤ الوسيطي المشروط يعمل بشكل جيد مقارنة بالتنبؤات الأخرى. تم ايجاد التنبؤ الوسيطي الاعظم غير المتحيز ومقارنته مع باقي التنبؤات ووجدناه اكثر فعالية مــن التنبــؤ الوسيطي الاعظم تحت مقياس تقارب بيتمان.

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